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STRUCTURE OF THE GROUP OF SYMPLECTIC MATRICES AND THE STRUCTURE OF THE SET OF UNSTABLE CANONICAL SYSTEMS WITH PERIODIC COEFFICIENTS

V.A. Yakubovich

Translation of "Stroyeniye gruppy simplekticheskikh matrits i struktura mnozhestva neustoychivykh kanonicheskikh sistem s periodicheskimi koeffitsiyentami," Matematicheskiy sbornik, Vol. 44 (86), No. 3, 1958, pp. 313-352

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16. Abstract
The author derives necessary and sufficient conditions for the stability and instability of the solutions of a system of 2k linear differential equations with periodic coefficients in canonical form, in which the coefficients are functions of certain "structural" parameters which must be selected in such a way that the solutions satisfy certain bounds. The study is a generalization to higher order systems of the work of I. N. Gel'fand and V.B. Lidskiy and G. Kreyn for the case of a two-equations system. The instability and stability criteria are obtained by examining the structure of the coefficient matrix of the set of all unstable systems and the set of all systems whose solutions satisfy certain bounds.

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STRUCTURE OF THE GROUP OF SYMPLECTIC MATRICES AND THE STRUCTURE OF THE SET OF UNSTABLE CANONICAL SYSTEMS WITH PERIODIC COEFFICIENTS

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Introduction

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We will consider a system of 2k linear differential equations with periodic coefficients in canonical form

$$\frac{dx}{dt} = IH(t)x, \tag{0.1}$$

where H(t) is a symmetric matrix of piecewise continuous real periodic functions with period 1,

$$I = \begin{pmatrix} 0 & -E_k \\ E_k & 0 \end{pmatrix},$$

E_k is the unit matrix of order k, and x is a vector. Such systems, which are very important in applications, were studied by various authors [1-12] and others. The basic results were obtained by M. G. Kreyn [1-6].

The problems which are encountered in applications have the following character. The coefficients of system (0.1) are functions of certain "structural" parameters. They must be selected in such a way that all solutions of system (0.1) are bounded as $t \to \infty$ (the corresponding motion is stable) or in such a way that among the solutions there are unbounded solutions (unstable motion), or in such a way that the solutions satisfy the inequality

$$\|x(t)\| < Ce^{\alpha t}, \quad t \to \infty \tag{0.2}$$

for a given $\alpha > 0$, etc. Sometimes, it is required to construct in parameter space the corresponding regions or at least clarify where these regions lie.

Sometimes the coefficients in system (0.1) (all or some of them) are not known exactly, for example only their upper and lower bounds

*Numbers in the margin indicate pagination in the foreign text.

are known. Certain conclusions must be made with regard to the boundedness or unboundedness of the solutions or the relative rate of increase of the solutions as $t \to \infty$.

We note that such bounds present certain difficulties even in systems with constant coefficients, when the system is integrated in explicit form (if we have in mind efficient solutions).

The stability and instability criteria, and the estimates of /314
the characteristic exponent give, to some extent, an answer to the problems posed. At the present time, there are many known stability criteria for system (0.1) (efficient and "exact" sufficient stability conditions), but there are considerably fewer instability criteria, and practically no estimates for the characteristic exponents.

In order to compare the various criteria and to understand them, it is advantageous to study from the above point of view (boundedness, unboundedness, etc. of the solutions) the entire set of systems (0.1). This is also advantageous because the parameters may enter the coefficients of system (0.1) in various ways. By studying the functional space $\mathfrak{H} = \{H(t)\}$ we can reach certain conclusions about the stability and instability regions in parameter space in each concrete case. Such a study was undertaken in the work of I.M. Gel'fand and V.B. Lidskiy [7]. It was shown in Ref. 7 that the set of all "strongly stable" matrices $H(t)$ decomposes in $\mathfrak{H}$ into a denumerable number of regions, and it is explained by which properties of the solutions the systems (0.1) are characterized in a particular stability region.

Our main purpose will be to study the structure of the set of all unstable systems (0.1), and also the set of systems whose solutions satisfy the bound (0.2). In fact, we solve a much more general problem, which is related to the study of the group of real simplectic matrices, a problem which is formulated at the end of the introduction. Having solved this problem, we will be able to answer questions of the following type: What is the structure of the set of matrices $H(t)$ for which the systems (0.1) have j, $0 \leqslant j \leqslant k$ linearly independent solutions with characteristic exponents α in the interval

$0 \leqslant \alpha_0 < \alpha < \alpha_1$ where the numbers α_0, α_1 are given, and j linearly[1] independent solutions with characteristic exponents $-\alpha$, $-\alpha_1 < -\alpha < -\alpha_0$ and 2k - 2j linearly independent solutions which are bounded both as $t \to \infty$ and as $t \to -\infty$? It is possible to define for the first j characteristic exponents various bounds, and to impose certain conditions on the bounds of the solution, etc.

By studying the structure of this set of "stable" and "unstable" systems (0.1), we will arrive naturally at a general method which will be used to obtain sufficient stability and instability conditions, a topic which is the subject matter of this article.

We will use in the discussion which follows the following notation:

U^τ transpose of matrix

U^* transposed complex-conjugate matrix,

E unit matrix,

if H is a real symmetric or Hermitian matrix ($H^* = H$), then /315

$$\|H\| = \max_j |h_j|,$$

where h_j are the eigenvalues of the matrix H, and for an arbitrary matrix U

$$\|U\| = \sup_x \frac{\|Ux\|}{\|x\|}.$$

For real symmetric or Hermitian matrices H_1, H_2, the inequality $H_1 \leqslant H_2$ denotes that $(H_1 a, a) \leqslant (H_2 a, a)$ for all vectors a.

We will state the fundamental definitions and assumptions mainly due to M.G. Kreyn, which will be used below.

I. The matrix X(t) of the fundamental system of solutions of system (0.1) determined from the condition X(0) = E is called the matricant of system (0.1). The value of the matricant in the period

[1]This condition follows from the previous condition, since system (0.1) has for the solution with characteristic exponent α also a solution with characteristic exponent $-\alpha$.

X(1) is called the monodromy matrix of system (0.1). The eigenvalues of the monodromy matrix are called the multipliers of system (0.1).

II. The matrix X(t) for any t is simplectic, i.e., it satisfies the relation $X^*IX = I$. The group of all real simplectic matrices will be denoted by $\mathfrak{S}$. The spectrum of a simplectic matrix is symmetric with respect to the real axis and with respect to the unit sphere, i.e. the eigenvalues of a simplectic matrix decompose into the pairs $e^{i\phi}$, $e^{-i\phi}$, the pairs μ and μ^{-1}, and the quadruples $re^{i\phi}$, $re^{-i\phi}$, $r^{-1}e^{i\phi}$, $r^{-1}e^{-i\phi}$(r, ϕ, μ are real). The multiplicity of the eigenvalues which are equal to 1 and -1 is necessarily even.

III. The eigenvalues of a simplectic matrix X which lie on the unit circle are divided into eigenvalues of the first and second kind.

Let $\rho = e^{i\phi}$ be a simple eigenvalue of the matrix X and a be the corresponding eigenvector $Xa = e^{i\phi}a$. Then [5],$(1/i)(Ia,a)$ is a real number which is different from zero. The eigenvalue $\rho = e^{i\phi}$ is called an eigenvalue of the first kind if $(1/i)(Ia,a) > 0$, and of the second kind, if $(1/i)(Ia,a) < 0$. From the above, we can easily derive that the eigenvalues $\rho = e^{i\phi}$ and $\rho^* = e^{-i\phi}$ are of opposite kinds.

Let $\rho = e^{i\phi}$ be an eigenvalue of multiplicity m and $\mathcal{L}_\rho$ the corresponding subspace of roots. We assume that on $\mathcal{L}_\rho$ the Hermitian form $(1/i)(Ix,x)$ is diagonalized with m_1 positive and m_2 negative squares. On $\mathcal{L}_\rho$, the form $(1/i)(Ix,x)$ is non-singular, therefore $m_1 + m_2 = m$. In this case, we say that m_1 eigenvalues of the first kind and m_2 eigenvalues of the second kind coincide. In particular, if in $\mathcal{L}_\rho$ /316
the form $(1/i)(Ix,x)$ is positive (negative) definite, then $\rho = e^{i\phi}$ is an eigenvalue of the first (second) kind of multiplicity m. In the last case, simple elementary divisors n of matrix $X-\lambda E$ correspond to the eigenvalue ρ, i.e. in the invariant subspace $\mathcal{L}_\rho$ the matrix X is reduced to diagonal form.

If at the point $\rho = e^{i\phi}$, m_1 eigenvalues of the first kind and m_2 eigenvalues of the second kind coincide, then at the point $\rho^* = e^{-i\phi}$,

m_2 eigenvalues of the first kind and m_1 eigenvalues of the second kind coincide.

Below, it will also be convenient to consider eigenvalues which lie in the interior of the unit circle in terms of eigenvalues of the first kind and those which lie outside this circle in terms of eigenvalues of the second kind. Thus, the simplectic matrix X has k eigenvalues of the first kind and k eigenvalues of the second kind.

The multipliers, like the eigenvalues of the monodromy matrix, are also classified into multipliers of the first and second kind. The multipliers can also be defined equivalently in another way (M. G. Kreyn [5], Par. 4, I.M. Gel'fand and V.B. Lidskiy [7], p. 7).

IV. Let γ be a closed contour in the complex plane which is symmetric with respect to the unit circle which does not pass through the eigenvalues of the simplectic matrix X. Let $\mathscr{L}_0$ be a subspace of dimension m which is the union of the subspaces which correspond to the eigenvalues which lie in the contour γ. The subspace $\mathscr{L}_0$ will be nondegenerate under the metric $(1/i)(Ix,x)$. Let P be its projection matrix. The number m_1 eigenvalues of the first kind and the number m_2 eigenvalues of the second kind which lie on the contour γ are equal, respectively, to the number of positive and negative eigenvalues of the Hermitian matrix

$$\frac{1}{i} P^* I P = \frac{1}{i} I P = \frac{1}{i} P^* I. \tag{0.3}$$

The matrix P can be defined using the formula derived by M.G. Kreyn:

$$P = \frac{1}{2\pi i} \oint_{\gamma} (\rho E - X)^{-1} d\rho. \tag{0.4}$$

Instead of the contour γ, any contour can be used which contains part of the spectrum and which is symmetric with respect to the unit sphere.

V. System (0.1) is said to be stable if all its solutions are bounded as $t \to \infty$. If, in addition to this, all systems with matrices $H_1(t)$ sufficiently close to $H(t)$ have this property, then system (0.1) is said to be strongly stable.[2]

System (0.1) is said to be unstable if among its solutions /317
solutions exist which are not bounded as $t \to \infty$, and strongly unstable if this is also valid for all systems with matrices $H_1(t)$ which are sufficiently close to $H(t)$.

A necessary and sufficient condition that system (0.1) be stable is that all its multipliers lie on the unit circle and that they correspond to simple elementary divisors of the matrix.

A necessary and sufficient condition that system (0.1) be stable is that all its multipliers lie on the unit circle and that among them there be no repeated multipliers of various kinds [5-7].

A necessary and sufficient condition that system (0.1) be strongly stable is that at least one of its multipliers do not lie on the unit circle.

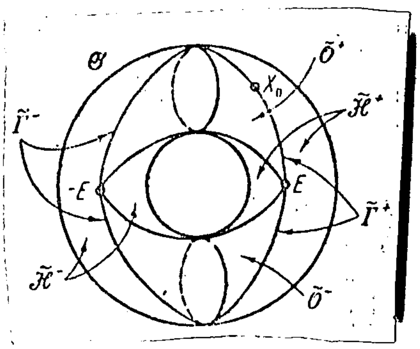

Fig. 1.

We will consider the case of a system of equations of order greater than two, k > 1 (of course, the discussion is also valid for k = 1). The case of a system consisting of two equations has been studied in sufficient detail from various points of view [2, 4, 5, 8, 9-12]. For the case k = 1, the entire presentation which follows can be made very concise (see [7], Par. 9, and [12], Par. 3). The group $\mathfrak{G}$ for k = 1 is homeomorphic to the interior of a torus (see Fig. 1). The matricant X(t) is a trajectory in $\mathfrak{G}$, which starts at

[2]Closeness is understood in the sense of the distance

$$r(H_1, H_2) = \int_0^1 \| H_1(t) - H_2(t) \| \, dt.$$

This definition is due to I.M. Gel'fand.

the point E; conversely, any (piecewise smooth) trajectory defines some system (0.1). The stability or instability of system (0.1) is only determined by the monodromy matrix, i.e. the end of the trajectory X(t). All strongly stable ends X(1) lie in the regions $\tilde{\mathcal{O}}^{+}$ and $\tilde{\mathcal{O}}^{-}$, and all unstable ends X(1) in the region $\tilde{\mathcal{H}}^{+}, \tilde{\mathcal{H}}^{-}$. The common boundaries of these regions (the "cones" $\tilde{\Gamma}^{+}$ and $\tilde{\Gamma}^{-}$) consist of matrices for which the multipliers of the first and second kind coincide (for k = 1, this can only happen at the points $\rho = 1$ and $\rho = -1$). The set of strongly stable (strongly unstable) trajectories X(t) decomposes into classes of trajectories which can be deformed into one another continuously without displacing the end from the corresponding region $\tilde{\mathcal{O}}^{+}, \tilde{\mathcal{O}}^{-}$ (or $\tilde{\mathcal{H}}^{+}, \tilde{\mathcal{H}}^{-}$). These trajectories form the stability and instability regions in the functional space $\mathfrak{H} = \{H(t)\}$.

Our problem will be to clarify the analogous picture in the case k > 1. We note once more that the structure of the stability regions has been studied in [7]. In Ref. 7, it was also shown that in the general case the group $\mathfrak{G}$ is "similar" to the interior of a torus, it is homeomorphic to the topological product of the circle with the connected and simply connected topological space.

The difficulties which arise when the set of unstable systems (0.1) is studied are the following. In the stable case, the monodromy matrix is always reduced to diagonal form; in the unstable case, the monodromy matrix may have a complex canonical structure. /318
We will consider, as was done in Ref. 7, the curves in the group $\mathfrak{G}$ and the deformations of these curves. In the unstable case, the canonical structure of the monodromy matrices along such curves can change in a variety of ways. This makes all proofs extremely complex and forces us, in contrast to Ref. 7, to develop a certain formal apparatus. In particular, we will introduce the topological space Σ, in which, roughly speaking, the point ζ is the set of all eigenvalues of the matrix X (taking into consideration the kind of eigenvalue), and we will determine the properties of the mapping $\zeta = \zeta(X)$.

The main problem which we will solve in this article is as follows. Let $\mathfrak{M}$ be some region in Σ, $\tilde{\mathfrak{M}}$ its image in $\mathfrak{G}$, and $\mathfrak{M}$

a set of matrices H(t) for which Eq. (0.1) has a monodromy matrix in $\tilde{\mathfrak{M}}$. We must clarify the structure of the sets $\tilde{\mathfrak{M}}$ and $\mathfrak{M}$. We will show that $\tilde{\mathfrak{M}}$ is always a domain, and $\mathfrak{M}$ decomposes into a finite or countable number of regions which depend on certain properties of the set $\tilde{\mathfrak{M}}$ (see below, Theorem 4.2, Par. 4). In other words, our main problem will be the study of the structure of the set of systems (0.1), whose multipliers have certain given properties.

1. Fundamental Definitions and Lemmas

1. We denote by $\mathfrak{H} = \{H(t)\}$ the linear space of matrices in (0.1), with norm $\int_0^1 \| H(t) \| dt$, by $\mathscr{H}$ the set of matrices H(t) which correspond to the strongly unstable systems (0.1), by $\mathcal{O}$, the set of matrices H(t) which correspond to the strongly stable systems (0.1), and by Γ_0, their common boundary.

$\mathcal{O}$ and $\mathscr{H}$ are open sets and $\mathfrak{H} = \mathcal{O} \cup \Gamma_0 \cup \mathscr{H}$.

Let X(t), $0 \leqslant t \leqslant 1$ be the trajectory in the group of real simplectic matrices $\mathfrak{S}$, which begins at E, X(0) = E, which has the property that dX/dt is a piecewise continuous function of t. The set of all such trajectories will be denoted by $\mathfrak{S}(t)$.

To each matrix $H(t) \in \mathfrak{H}$ corresponds the matricant $X(t) \in \mathfrak{S}(t)$ of the equation (0.1). The converse proposition is also valid ([7], Lemma 1, p. 12). This correspondence will be a homeomorphism if in the set $\mathfrak{S}(t)$ distance is defined according to the formula

$$r(X_1, X_2) = \int_0^1 \| X_1(t) - X_2(t) \| dt + \int_0^1 \left\| \frac{dX_1}{dt} - \frac{dX_2}{dt} \right\| dt$$

(see, for example, [12], p. 38-39).

Sets corresponding in $\mathfrak{S}(t)$ to the sets $\mathcal{O}, \mathscr{H}, \Gamma_0$, will be denoted by the same letters.

It follows from Par. V in the introduction that only the end of the trajectory $X(1) \in \mathfrak{S}$ determines whether X(t) belongs to the sets $\mathcal{O}, \mathscr{H}, \Gamma_0$. The corresponding sets in $\mathfrak{S}$ will be denoted by /319
$\tilde{\mathcal{O}}, \tilde{\mathscr{H}}, \tilde{\Gamma}_0$ (see also the table below). Thus, $X(t) \in \mathcal{O}$, if $X(1) \in \tilde{\mathcal{O}}$, etc.

Let $\rho_1, \ldots, \rho_k$ be the eigenvalues of the first kind of the matrix . Then the numbers $\rho_1^{-1}, \ldots, \rho_k^{-1}$ will be its eigenvalues of the second kind.

The symbol $\zeta = \zeta(X)$ will denote the set of all eigenvalues of the matrix X taking into account their kind. The set of all similar ζ will be denoted by Σ. In other words, an element ζ of the set Σ is the set of 2k complex numbers $\rho_1, \ldots \rho_k$ (multipliers of the first kind) and $\rho_1^{-1}, \ldots, \rho_k^{-1}$ (multipliers of the second kind) which satisfy the conditions:

1) $0 < |\rho_j| \leqslant 1$ (for multipliers of the first kind);

2) if ρ_j is a multiplier of the first kind in ζ and $|\rho_j| < 1$, then also $\rho_j^* \in \zeta$.

Thus, on the intervals (-1, 0) and (0, 1) of the real axis and on the unit circle, the multipliers of the first kind can be in arbitrary positions, but in the region $|\rho|<1, \mathcal{J}\rho\neq 0$ they are symmetric with respect to the real axis.

We will write:

$$\zeta = \{\rho_1, \ldots, \rho_k;\ \rho_1^{-1}, \ldots, \rho_k^{-1}\},$$

entering first multipliers of the first kind, and then multipliers of the second kind. The order in which the multipliers of the first kind are written is immaterial: The elements

$$\zeta_1 = \{\rho_1', \ldots, \rho_k';\ (\rho_1')^{-1}, \ldots, (\rho_k')^{-1}\} \text{ and } \zeta_2 = \{\rho_1'', \ldots, \rho_k'';\ (\rho_1'')^{-1}, \ldots, (\rho_k'')^{-1}\}$$

are considered identical if a substitution

$$\begin{pmatrix} 1, 2, \ldots, k \\ s_1, s_2, \ldots, s_k \end{pmatrix}$$

exists, such that $\rho_j' = \rho_{s_j}''$ $(j = 1, 2, \ldots, k)$.

We introduce a natural topology in Σ: an ε-neighborhood of the point $\zeta_0 = \{\rho_1^0, \ldots, \rho_k^0; \ldots\}$ will be the set of all $\zeta = \{\rho_1, \ldots, \rho_k; \ldots\} \in \Sigma$ such that $|\rho_j - \rho_{s_j}^0| < \varepsilon$, where $s_1, \ldots, s_k$ is an arbitrary permutation of the numbers 1, ..., k. Σ becomes then a topological space.

From the "identity" multipliers of the first (and consequently the second) kind which were introduced above, we have the following: If $\zeta(t)=\{\rho_1(t),\dots,\rho_k(t);\dots\}, 0\leq t\leq 1$ is a closed curve in the space Σ, the $\rho_j(t)$ are not necessarily closed curves in the complex plane. Only $\rho_j(1)=\rho_{s_j}(0)$ will hold where $s_1, \dots, s_k$ is a permutation of the numbers 1, ..., k. We must adopt a definition of the symbol ζ, i.e., have "identity" multipliers of one kind, such that if the matrix X describes a closed curve in the group $\mathfrak{G}$, then in general /320 its eigenvalues do not describe closed curves in the complex plane.

We will adopt the following notation. Let $\mathfrak{M}$ be a set in $\mathfrak{H}$. The set of corresponding monodromy matrices will be denoted by $\tilde{\mathfrak{M}}$, $\tilde{\mathfrak{M}}\subset\mathfrak{G}$. The corresponding set in Σ will be denoted by $\hat{\mathfrak{M}}$, $\hat{\mathfrak{M}}=\zeta(\tilde{\mathfrak{M}})$.

In addition to the sets $\tilde{\mathcal{O}}, \tilde{\mathcal{H}}, \tilde{\Gamma}_0, \hat{\mathcal{O}}, \hat{\mathcal{H}}, \hat{\Gamma}_0$, we will consider the sets $m_\alpha, \tilde{m}_\alpha, \hat{m}_\alpha, \alpha>0,$ corresponding to the systems (0.1), whose solutions satisfy the bound (0.2), i.e. whose multipliers lie in the interior of the circle with radius e^α, and the sets $M_\alpha, \tilde{M}_\alpha, \hat{M}_\alpha$ corresponding to the systems (0.1), for which at least one multiplier lies outside the circle of radius $e^\alpha > 1$ (the solutions of these systems do not satisfy the bound (0.2)).

We summarize the notation which was introduced in the table:

$\mathfrak{H}=\{H(t)\}$	$\mathfrak{G}(t)=\{X(t)\}$	$\mathfrak{G}=\{X\}$	$\Sigma=\{\zeta\}$	
$\mathcal{O}$	$\mathcal{O}$	$\tilde{\mathcal{O}}$	$\hat{\mathcal{O}}$	All multipliers lie on the unit circle and there are no repeated multipliers of different kinds (strong stability).
$\mathcal{H}$	$\mathcal{H}$	$\tilde{\mathcal{H}}$	$\hat{\mathcal{H}}$	Multipliers exist which do not lie on the unit circle (strong instability)
Γ_0	Γ_0	$\tilde{\Gamma}_0$	$\hat{\Gamma}_0$	All multipliers lie on the unit circle; there are no repeated multipliers of different kinds.
Γ	Γ	$\tilde{\Gamma}$	$\hat{\Gamma}$	Among the multipliers, there are repeated multipliers of different kinds.

[Table continued on following page.]

m_α	m_α	$\tilde{m}_\alpha$	$\hat{m}_\alpha$	All multipliers lie in the interior of the circle with radius $e^\alpha > 1$ (bound (0.2) holds)
M_α	M_α	$\tilde{M}$	$\hat{M}_\alpha$	Among the multipliers, there are multipliers which lie in the exterior of the circle with radius $e^\alpha > 1$ (estimate (0.2) does not hold

This table also contains the set $\Gamma(\tilde{\Gamma}, \hat{\Gamma})$, which we will encounter below.

2. Below, we will need certain comparatively sophisticated properties of the mapping $\zeta = \zeta(X)$ (see Par. 2, Theorem 2.4) which we could only prove by using a canonical decomposition of the real simplectic matrices X. Before we state the theorem which formulates this decomposition, we will state a theorem about the canonical decomposition of matrices of the form IH, where I is a skew symmetric non-singular matrix, and H is a symmetric real matrix, since this theorem is related directly to canonical systems of the type (0.1). Both theorems will be stated without proof, since their proofs differ only slightly from the proofs for analogous propositions (see A.I. Mal'tsev [13], p. 346-417).

We will consider a real simplectic space $\mathfrak{E}$, i.e. a real vector space of dimension 2k, with the inner product /321

$$\langle x, y \rangle = (Gx, y) = \sum_{i,j} g_{ij} x_i y_j,$$

where $G = \| g_{ij} \|$ is a non-singular real skew symmetric matrix.

The operators K, whose matrices have, in the given basis, the form

$$K = G^{-1}H$$

where $H^\tau = H$ is a real symmetric matrix, are skew symmetric operators in the space $\mathfrak{E}$:

$$\langle Kx, y \rangle = (Hx, y) = -\langle x, Ky \rangle.$$

Below, we will always denote by $Q_\varepsilon(\alpha)$ the Jordan form:

$$Q_\varepsilon(\alpha)=\begin{pmatrix} \alpha & \varepsilon & 0 & 0 & \dots & 0 \\ 0 & \alpha & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \alpha & \varepsilon & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & 0 & \dots & \varepsilon \\ 0 & 0 & 0 & 0 & \dots & \alpha \end{pmatrix}.$$

Theorem 1.1.1^0. The elementary divisors of the matrix $G^{-1}H-\lambda E$ may be of four different types:

I) elementary divisors of the form λ^{2m},

II) the pairs $(\lambda-\lambda_0)^m$, $(\lambda+\lambda_0)^m$ with λ_0 real,

III) the pairs $(\lambda+i\varphi)^m$, $(\lambda-i\varphi)^m$; $\varphi\neq 0$

IV) the fourtuples $(\lambda\pm\alpha\pm i\beta)^m$, $\alpha\neq 0$, $\beta\neq 0$ (m is an integer, ϕ, α, β are real numbers).

2^0. To each elementary divisor of type I) or to the pairs of elementary divisors of type II), III) or a fourtuple of the form IV) corresponds a certain subspace which is invariant with respect to the operator K. All these subspaces are orthogonal and nondegenerate under the metric <x, y>. The entire space $\mathfrak{E}$ can be decomposed into under the direct sum of these subspaces.

3^0. In each subspace $\mathfrak{E}_j$, a basis can be chosen, in which the matrix K_j of the operator K and the Gramm matrix G_j have the following form:

$$\text{I) } K_j=Q_\varepsilon(0),\quad G_j=\begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

$$\text{II) } K_j=\begin{pmatrix} Q\ (\lambda_0) & 0 \\ 0 & -Q_\varepsilon(\lambda_0)^\tau \end{pmatrix},\quad G_j=\pm\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

respectively, where the sign for G_j can be chosen arbitrarily; /322

$$\text{III) } K_j=\begin{pmatrix} 0 & -Q_\varepsilon(\varphi) \\ Q_\varepsilon(\varphi) & 0 \end{pmatrix},\quad G_j=\varepsilon_0\begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix},\quad N=\begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \dots & 0 & 0 \end{pmatrix},$$

$\varepsilon_0 = 1$ or -1, which is determined by the properties of the operator K. By selecting appropriately the sign of ϕ, we can have $\varepsilon_0 = 1$

IV) $K_j = \begin{pmatrix} P & 0 \\ 0 & -P^{\tau} \end{pmatrix}$, where $P = \begin{pmatrix} Q_{\varepsilon}(\alpha) & \beta E \\ -\beta E & Q_{\varepsilon}(\alpha) \end{pmatrix}$, $G = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$

(here, P, E are matrices of order 2m, K_j, G_j are matrices of order 4m).

To a change of basis

$$f_i = \sum_j s_{ij} e_j$$

($S = \| s_{ij} \|$ is an arbitrary non-singular real matrix) corresponds a transformation of the Gramm matrix $G = \| \langle e_i, e_j \rangle \|$ and the matrix of the operator K:

$$G_1 = T^{\tau} G T, \; K_1 = T^{-1} K T$$

where $T = S^{\tau}$. Therefore, Theorem 1.1 can be stated as follows:

There exists a real non-singular matrix T, such that the matrices $T^{-1}G^{-1}HT$ and $T^{\tau}GT$ can be factored into matrices K_j, G_j having the form shown above:

$$T^{-1}G^{-1}HT = K_1 \oplus K_2 \oplus \ldots,$$
$$T^{\tau}GT = G_1 \oplus G_2 \oplus \ldots.$$

If the system
$$\frac{dx}{dt} = G^{-1}Hx,$$

where G, H are real, $\det G \neq 0$, $G^{\tau} = -G$, $H^{\tau} = H$, is a canonical system, heorem 1.1 states that such system with constant coefficients can be reduced by the appropriate transformation

$$x = Ty$$

with a real non-singular matrix T to a system which can be decomposed into a number of canonical systems

$$\frac{dy_j}{dt} = K_j y_j, \quad K_j = G_j^{-1} H_j,$$

where the matrices K_j and G_j have the form which was stated in the theorem. In particular, system (0.1) with constant coefficients can be transformed in this manner (for system (0.1), $G = I^{-1} = -I$).

We will associate with a subspace of type III) the elementary divisors $(\lambda \pm i\varphi)^m$, if $\varepsilon_0 = 1$, and $-(\lambda \pm i\varphi)^m$, if $\varepsilon_0 = -1$ and $\phi > 0$. The system of elementary divisors in which the elementary divisors $(\lambda \pm i\varphi)^m$, $\phi > 0$ have a definite sign, will be referred to as the system of elementary divisors for which the sign is defined (see A.I. Mal'tsev [13], Par. 4). Thus, the structure of the canonical decomposition of the matrix $G^{-1}H$ is completely determined by the system of elementary divisors for which the sign has been defined. /323

The operator X is said to be symplectic if for all $a, b \in \mathfrak{E}$

$$\langle Xa, Xb \rangle = \langle a, b \rangle .$$

The matrix X of the symplectic operator in a basis where G is the Gramm matrix which satisfies the relation

$$X^\tau G X = G \qquad (1.1)$$

is called G-orthogonal.[3] In particular, if $G = I^{-1} = -I$, this relation becomes

$$X^\tau I X = I$$

i.e. X is a simplectic or I-orthogonal matrix, $X \in \mathfrak{S}$.

It can be easily verified that the matrices $\pm e^{G^{-1}H}$ satisfy the relation (1.1), if $H^\tau = H$.

Theorem 1.2.1^0. The matrices of the symplectic operators can have four types of elementary divisors:

I) $(\lambda - 1)^{2m}$ or $(\lambda + 1)^{2m}$;

II) the pairs $(\lambda - \mu_0)^m$, $(\lambda - \mu_0^{-1})^m$ with μ_0 real,

III) the pairs $(\lambda - e^{i\varphi})^m$, $(\lambda + e^{i\varphi})^m$;

IV) the fourtuples $(\lambda - re^{\pm i\varphi})^m$, $(\lambda - r^{-1}e^{\pm i\varphi})^m$ (m is an integer, ϕ, $r > 0$ are real numbers).

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[3]Only an I-orthogonal matrix will be called a symplectic matrix.

2^0. To each elementary divisor of type I) or pairs of elementary divisors of type II), III) or each fourtuple of type IV) corresponds a subspace which is invariant with respect to the operator X. All these subspaces are orthogonal and nondegenerate under the metric <x, y> . The entire space $\mathfrak{E}$ can be decomposed into a direct sum of these subspaces.

3^0. In each subspace $\mathfrak{E}_j$, a basis can be selected in which the matrix X_j of the operator X and the Gramm matrix G_j have, respectively, the form: /324

$$\text{I) } X_j = \pm e^{Q_\varepsilon(0)}, \quad G_j = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & -1 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & -1 & \dots & 0 & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}$$

(a plus corresponds to the elementary divisor $(\lambda - 1)^m$ and a minus to the elementary divisor $(\lambda + 1)^m$;)

$$\text{II) } X_j = \operatorname{sign} \mu_0 \cdot e^{K_j},$$

where

$$K_j = \begin{pmatrix} Q_\varepsilon(\lambda_0) & 0 \\ 0 & -Q_\varepsilon(\lambda_0)^\tau \end{pmatrix}, \quad \lambda_0 = \ln|\mu_0|, \quad G_j = \pm \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

where the sign of G_j can be chosen arbitrarily;

$$\text{III) } X_j = \exp\begin{pmatrix} 0 & -Q_\varepsilon(\varphi) \\ Q_\varepsilon(\varphi) & 0 \end{pmatrix} = \begin{pmatrix} \cos Q_\varepsilon(\varphi) & -\sin Q_\varepsilon(\varphi) \\ \sin Q_\varepsilon(\varphi) & \cos Q_\varepsilon(\varphi) \end{pmatrix},$$

$$G_j = \varepsilon_0 \cdot \begin{pmatrix} 0 & N \\ -N & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \dots & 0 & 0 \end{pmatrix},$$

$\varepsilon_0 = \pm 1$ which is determined by the property of the transformation X. By selecting the sign of ϕ appropriately, we can have $\varepsilon_0 = 1$,

$$\text{IV) } X_j = \begin{pmatrix} e^P & 0 \\ 0 & e^{-P^\tau} \end{pmatrix}, \quad P = \begin{pmatrix} Q_\varepsilon(\alpha) & -\varphi E \\ \varphi E & Q_\varepsilon(\alpha) \end{pmatrix}, \quad G_j = \pm \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix},$$

$$\alpha = \ln r,$$

and the sign of G_j can be chosen arbitrarily.

It follows from Theorem 1.2, as before, that for any symplectic matrix X a real matrix T can be found such that the matrices $T^{-1}XT$ and $T^{T}GT$ factor simultaneously into the matrices X_j, G of the form given above:

$$T^{-1}XT = X_1 \oplus X_2 \oplus X_3 \oplus \ldots,$$
$$T^{\tau}GT = G_1 \oplus G_2 \oplus G_3 \oplus \ldots$$

In particular, if the matrix X is reduced to diagonal form, this decomposition has the form:

$$\left.\begin{array}{l} T^{-1}XT = \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_0^{-1} \end{pmatrix} \oplus e^{I_2\varphi} \oplus \begin{pmatrix} re^{I_2\varphi} & 0 \\ 0 & r^{-1}e^{I_2\varphi} \end{pmatrix} \oplus \ldots, \\ T^{\tau}GT = I_2^{-1} \oplus \varepsilon_0 I_2^{-1} \oplus \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \oplus \ldots; \end{array}\right\} \qquad (1.2)$$

Here, we wrote out the three possible factors and /325

$$I_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e^{I_2\varphi} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To the factor $e^{I_2\phi}$ corresponds an eigenvalue of the first kind $e^{i\phi}$ and an eigenvalue of the second kind $e^{-i\phi}$, if $\varepsilon_0 = 1$, and vice versa if $\varepsilon_0 = -1$. Therefore, if in decomposition (1.2) we denote by ϕ the argument of the multiplier of the first kind, then $\varepsilon_0 = 1$.

Remark. If we embed $\mathfrak{E}$ in a complex vector space, then the subspace of type III) can be decomposed into the direct sum of two complex cyclic subspaces which correspond to the eigenvalues $e^{i\phi}$ and $e^{-i\phi}$. In the first subspace, we can select a basis whose coordinate matrix has the form $\begin{pmatrix} E \\ -iE \end{pmatrix}$ (each column defines the coordinates of the vector. This basis is not cyclic but it is convenient in the sense that

$$e^{\begin{pmatrix} 0 & -Q_6(\varphi) \\ Q_6(\varphi) & 0 \end{pmatrix}} \cdot \begin{pmatrix} E \\ -iE \end{pmatrix} = \begin{pmatrix} E \\ -iE \end{pmatrix} \cdot e^{iQ_6(\varphi)}.$$

The Gramm matrix of the form $\frac{1}{i}(Ix, x)=\frac{1}{i}(G^{-1}x, x)$ has in this basis the form:

$$\frac{1}{i}\varepsilon_0\cdot\begin{pmatrix} E \\ -iE\end{pmatrix}^{*}\cdot\begin{pmatrix} 0 & -N \\ N & 0\end{pmatrix}\cdot\begin{pmatrix} E \\ -iE\end{pmatrix}=2\varepsilon_0 N.$$

Let us denote by m the dimension of the subspace under consideration (the dimension of the matrix N). If $m = 2m_1$ is even, then N has m_1 eigenvalues which are equal to 1 and m_1 eigenvalues which are equal to -1. Thus, in this case, regardless of ε_0, m_1 eigenvalues of the first kind and m_1 eigenvalues of the second kind will coincide at the point $\rho = e^{i\phi}$. The same holds also for the point $\rho = e^{-i\phi}$, except that the corresponding coordinate matrix has the form $\begin{pmatrix} E \\ iE\end{pmatrix}$.

If $m = 2m_1 + 1$ is odd, N has $m_1 + 1$ eigenvalues which are equal to 1, and m_1 eigenvalues which are equal to -1. Thus, in this case, when $\varepsilon_0 = 1$, $m_1 + 1$ eigenvalues of the first kind and m_1 eigenvalues of the second kind coincide at the point $\rho = e^{i\phi}$. Conversely, at the point $\rho = e^{-i\phi}$ $m_1 + 1$ eigenvalues of the second kind and m_1 eigenvalues of the first kind coincide. The converse is true when $\varepsilon_0 = -1$.

We will prove that by choosing appropriately the sign of ϕ we can always have $\varepsilon_0 = 1$. In fact, if $\varepsilon_0 = -1$, we must make a change of basis in the subspace of type III), which leads to the transformations

$$T^{-1}\cdot\exp\begin{pmatrix} 0 & -Q_\varepsilon(\varphi) \\ Q_\varepsilon(\varphi) & 0\end{pmatrix}\cdot T=\exp\begin{pmatrix} 0 & Q_\varepsilon(\varphi) \\ -Q_\varepsilon(\varphi) & 0\end{pmatrix},$$ /326

$$T^{*}\cdot\begin{pmatrix} 0 & N \\ -N & 0\end{pmatrix}T=\begin{pmatrix} 0 & -N \\ N & 0\end{pmatrix}.$$

It suffices if we take $$T=\begin{pmatrix} E & 0 \\ 0 & -E\end{pmatrix}.$$

We will state certain corollaries of Theorem 1.2.

1. To each $\zeta \in \Sigma|$ corresponds a matrix $X \in \mathfrak{G}|$ such that $\zeta = \zeta(X)$. For X, we can always take a matrix which was reduced to diagonal form.

We denote the right members of the decompositions (1.2) by X' and G'. For the given ζ, we select the corresponding matrix X'. We must prove that for an appropriate choice of the matrix T the matrix $X = TX'T^{-1}$ will belong to $\mathfrak{G}|$. No matter what the real non-singular skew symmetric matrices G' and G'', we can always find a real non-singular matrix T, such that $G'' = T^{\tau}G'T$. The latter follows from the fact that all spaces of the same dimension with a non-singular skew symmetric metric which have the same field for the coefficients are isomorphic. (See A.I. Mal'tsev [13], p. 349). Therefore, we can find a non-singular real matrix T such that $T^{\tau}I^{-1}T = G'$. Since the matrix X' is G'-orthogonal, $X'^{\tau}G'X' = G'$, the matrix X will be I^{-1}-orthogonal, i.e. $X \in \mathfrak{G}|$, and by construction $\zeta(X) = \zeta$.

2. The group $\mathfrak{G}|$ is connected.

This statement is proved simply in [7], but it can also be derived from the canonical decomposition. In fact, every matrix X_j can be connected continuously with the unit matrix without changing its structure. Then the matrix $T^{-1}XT$, and consequently also X are connected continuously with the unit matrix. This means that the group $\mathfrak{G}|$ is connected.

In the work of I.M. Gel'fand and V.B. Lidskiy [7], the important concept of an argument of a simplectic matrix was introduced.[4]

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[4]We will state this definition for the convenience of the reader. Any real non-singular matrix X can be represented in the form $X = SU$, where S is a real symmetric positive-definite matrix, and U is an orthogonal matrix. If X is a simplectic matrix, the matrices S and U will also be simplectic. An orthogonal simplectic matrix U of order 2k can be written in the form

$$U = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$$

where u_1, u_2 are matrices of order k. The matrix $w = u_1 + iu_2$ is unitary, $w \cdot w^* = E_k$. Therefore, $|\det w| = 1$, $\det w = e^{i\phi}$. The number ϕ is called the argument of the simplectic matrix X [7].

We will call the argument of $\zeta=\{\rho_1, \ldots, \rho_k; \rho_1^{-1}, \ldots, \rho_k^{-1}\}$ the sum /327
of the arguments of the multipliers of the first kind:

$$\operatorname{Arg}\zeta=\sum_{j=1}^{k}\operatorname{Arg}\rho_j.$$

In exactly the same way, the argument of the simplectic matrix X will be the sum of the arguments of its eigenvalues of the first kind. Thus,

$$\operatorname{Arg}X = \operatorname{Arg}\zeta(X)$$

Below (Par. 2, Theorem 2.1) we will prove, without using the concept of an argument, that $\zeta(X)$ is a continuous function of X. Thus, if X(t) is a continuous curve in the group $\mathfrak{G}$, we can renumber the multipliers of the first kind $\rho_j(t)$ so that $\rho_j(t)$ will be continuous functions. Thus, even though ArgX is a multivalued function,

$$\operatorname{Arg}X=(\operatorname{Arg}X)_0+2\pi m \quad (m=0, \pm 1, \pm 2, \ldots), \tag{1.3}$$

the increment in the argument

$$\operatorname{Arg}X(t)\big|_0^1=\operatorname{Arg}X(1)-\operatorname{Arg}X(0)$$

along any continuous curve is determined uniquely.

It follows from (1.3) that the increment in the argument along the closed curve X(t), $0 \leqslant t \leqslant 1$ is a multiple of 2π:

$$\operatorname{Arg}X(t)\big|_0^1=2\pi m \quad (m=0, \pm 1, \pm 2, \ldots).$$

Theorem 1.3. A necessary and sufficient condition that the curves $X_1(t)$ and $X_2(t)$, $0 \leqslant t \leqslant 1$ with common endpoints in the group $\mathfrak{G}$ be deformed[5] into one another without displacing the ends is:

[5] A deformation of the curve $X_1(t)$ into the curve $X_2(t)$, $0 \leqslant t \leqslant 1$ is a matrix of functions $X(t,s)$, $0 \leqslant t,s \leqslant 1$ which is continuous over the set t, s such that $X(t,0) = X_1(t,1) = X_2(t)$. Curves which have common endpoints and which can be deformed continuously one into another without displacing the endpoints will be called homotopic.

$$\operatorname{Arg} X_1(t)\,\Big|_0^1 = \operatorname{Arg} X_2(t)\,\Big|_0^1. \tag{1.4}$$

In particular, the closed curve $X(t)$, $0 \leqslant t \leqslant 1$ can be contracted into a point if and only if $\operatorname{Arg} X(t)\,\Big|_0^1 = 0.$

This statement was proved in the work of I.M. Gel'fand and V.B. Lidskiy [7] for the argument which was introduced in this paper. We denote by $\operatorname{Arg}_0 X$ the argument of the symplectic matrix X as defined in Ref. 7.[6] It suffices to show that equality (1.4) is equivalent to the equality

$$\operatorname{Arg}_0 X_1(t)\,\Big|_0^1 = \operatorname{Arg}_0 X_2(t)\,\Big|_0^1. \tag{1.5}$$

Let $X_1(0) = X_2(0) = X'$, $X_1(1) = X_2(1) = X''$, $X'pX''$ is the curve $X_1(t)$ and $X'qX''$ is the curve $X_2(t)$ (see Fig. 2). We denote by /328
$X(t)$, $0 \leqslant t \leqslant 1$ the curve $X'pX''qX'$

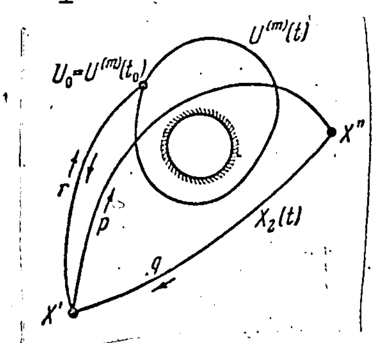

Fig. 2.

$$X(t) = \begin{cases} X_1(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ X_2(1-2t) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

The change in the argument along the curve $X(t)$ is a multiple of 2π. Let

$$\operatorname{Arg}_0 X(t)\,\Big|_0^1 = 2\pi m. \tag{1.6}$$

For any m closed curves $U^{(m)}(t)$, $0 \leqslant t \leqslant 1$ exist, such that

$$\operatorname{Arg} U^{(m)}(t)\,\Big|_0^1 = \operatorname{Arg}_0 U^{(m)}(t)\,\Big|_0^1 = 2\pi m. \tag{1.7}$$

In fact, this condition is satisfied, for example, for the curve ([7], p. 27):

[See following page.]

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[6] I.e., $\phi = \operatorname{Arg}_0 X$ (see reference on previous page).

$$U^{(m)}(t)=\left(\begin{array}{cccc|cccc} \cos 2\pi mt & & & & -\sin 2\pi mt & & & \\ & 1 & & & & 0 & & \\ & & \ddots & & & & \ddots & \\ & & & 1 & & & & 0 \\ \hline \sin 2\pi mt & & & & \cos 2\pi mt & & & \\ & 0 & & & & 1 & & \\ & & \ddots & & & & \ddots & \\ & & & 0 & & & & 0 \end{array}\right), \quad 0\leqslant t\leqslant 1 \tag{1.8}$$

(the empty entries have zeros).

We select an arbitrary point $U_0 = U^{(m)}(t_0)$ and we connect it by the curve U_0rX' with the point X' (this is possible, since the group $\mathfrak{G}$ is connected).

We denote by $V(t)$, $0 \leqslant t \leqslant 1$ the closed curve $U_0rX'pX''qX'rU_0$ (see Fig. 2). Clearly,

$$\mathrm{Arg}_0 V(t)\,|_0^1 = \mathrm{Arg}_0 X(t)\,|_0^1 = 2\pi m.$$

Thus, $\mathrm{Arg}_0 U^{(m)}(t)\,|_0^1 = \mathrm{Arg}_0 V(t)\,|_0^1$ and the curves $U^{(m)}(t)$ and $V(t)$ are homotopic. Since for a continuous deformation with fixed endpoints, the increment in the argument (Arg) does not change,

$$\mathrm{Arg}\, V(t)\,|_0^1 = \mathrm{Arg}\, U^{(m)}(t)\,|_0^1.$$

According to (1.7), $\mathrm{Arg}V(t) = 2\pi m$. Since $\mathrm{Arg}\, X(t)\,|_0^1 = \mathrm{Arg}\, V(t)\,|_0^1$,

$$\mathrm{Arg}\, X(t)\,|_0^1 = 2\pi m. \tag{1.9}$$

If condition (1.4) is satisfied, $m = 0$ in formula (1.9), and (1.5) follows from (1.6). This means that the curves can be deformed continuously into one another without displacing the endpoints. /329
If (1.4) is not satisfied, then in formula (1.9), and consequently in (1.6) $m \neq 0$, i.e. (1.5) is not satisified and the deformation can not be carried out. This proves the theorem.[7]

[7] We note that in addition to the usual properties of Arg X we used in this proof only equality (1.7). Therefore, Arg X can be defined in many ways which are equivalent in the sense that each definition will satisfy Theorem (1.3).

2. Properties of the Mapping $\zeta = \zeta(X)$

Theorem 2.1. The mapping $\zeta = \zeta(X)$ is continuous.[8]

Proof. We select an arbitrary matrix $X_0 \in \mathfrak{G}$, and we let

$$\zeta(X_0) = \zeta_0 = \{\rho_1, \ldots, \rho_k; \bar{\rho}_1^{-1}, \ldots, \bar{\rho}_k^{-1}\}.$$

We enclose all ρ_j, $|\rho_j| \neq 1$ by circles γ_j with radii which are so small that they do not intersect one another and the unit circle. All ρ_j on the unit circle are enclosed by circles which do not intersect with the circles which were drawn earlier and with one another. Let ρ_0 be a value of ρ_j and γ_0 be the corresponding circle. If the matrix X is sufficiently close to X_0, then the number of eigenvalues of the matrix X in the interior of γ_0 will be equal to the multiplicity of ρ_0. Suppose that $|\rho_0| = 1$, and that m_1 multipliers of the first kind and m_2 multipliers of the second kind coincided at the point ρ_0. We must prove that for the matrices X which are sufficiently close to X_0 there will also be m_1 multipliers of the first kind and m_2 multipliers of the second kind in the interior of γ_0.

We denote by P_0 the projection matrix of the subspace $\mathscr{L}_{\rho_0}$ of roots of the matrix X_0 and by P the projection matrix of the subspace which is the union of the root subspaces $\mathscr{L}_{\rho_j}$ of the eigenvalues $\rho_j \in \gamma_0$ of the matrix X. If the matrix X is sufficiently close to X_0, the matrix P is also close to P_0 (see Introduction, formula (0.4)). Then the number of positive and negative eigenvalues of the matrices $(1/i)IP_0$ and $(1/i)IP$ coincides, and is equal to, respectively, m_1 and m_2. This means that among the eigenvalues $\rho_j \in \gamma_0$ there are exactly m_1 of the first kind and m_2 of the second kind. The analogous statement is obvious for the eigenvalues which do not lie on the unit circle. Thus, for a sufficiently small neighborhood O_{ζ_0} (a certain set of circles which were constructed), and consequently also for any

[8] It is obvious that the eigenvalues of the matrix X are continuous functions of X. We must prove, roughly speaking, that the types of eigenvalues depend also continuously on X.

neighborhood O_ζ of the point ζ_0, we can find a neighborhood O_{X_0} of the point $X_0 \in \mathfrak{G}$, such that $\zeta(O_{X_0}) \subset O_{\zeta_0}$, which proves the theorem.

Theorem 2.2. The set of matrices $X \in \mathfrak{G}$, satisfying Eq. $\zeta(X) = \zeta_0$ is connected in the group $\mathfrak{G}$.[9]

Proof. Let $\zeta(X_1) = \zeta(X_2) = \zeta_0$. We will show that a continuous curve $X(t)$, $0 \leqslant t \leqslant 1$ exists which connects X_1 and X_2, which has the property that $\zeta|X(t)| \equiv \zeta_0$. /330

1. Both matrices X_1 and X_2 are reduced to diagonal form. Since for the matrices X_1 and X_2 the systems of eigenvalues and their kind coincide, in the canonical decomposition (1.2) the right members coincide for the matrices X_1 and X_2. (With the condition that in (1.2) ϕ denotes the argument of a multiplier of the first kind; then $\varepsilon_0 = 1$ and the matrix $G = I^{-1}$.) Thus,

$$T_1^{-1}X_1T_1 = T_2^{-1}X_2T_2, \quad T_1^{\tau}I^{-1}T_1 = T_2^{\tau}I^{-1}T_2.$$

Letting $U = T_1T_2^{-1}$, we have:[10]

$$X_2 = U^{-1}X_1U, \quad U^{\tau}IU = I. \tag{2.1}$$

The second relation shows that U is a simplectic matrix, $U \in \mathfrak{G}$.

Let X_1 and X_2 be arbitrary symplectic matrices which satisfy relation (2.1). We will show that the eigenvalues and their kind coincide for the matrices X_1 and X_2.

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[9]Here and below, connectedness is defined in the "narrow" sense: A set $\mathfrak{M} \subset \mathfrak{G}$ is said to be connected if any two points in $\mathfrak{M}$ can be connected in $\mathfrak{M}$ by a continuous curve.

[10]We also note that from Theorem (1.2) using the same arguments from which (2.1) was obtained, we can derive the following proposition: A necessary and sufficient condition that $X_2 = U^{-1}X_1U$, where X_1, X_2, U are simplectic matrices is that the systems of elementary divisors whose signs have been determined coincide for the matrices X_1 and X_2. (An analogous statement can be found in [13], p. 416.) It is easily seen that Theorem (1.2) can also be derived from this statement. In the subsequent proof, this theorem must be proved first. The same applies also to Theorem(1.1).

This is obvious for the eigenvalues ρ, $|\rho| \neq 1$. Let ρ be a simple eigenvalue of the matrix X_1, $|\rho| = 1$ and $X_1 a = \rho a$. We introduce the notation

$$b = Ua$$

Then, from (2.1), we have

$$X_2 b = \rho b \quad \text{and} \quad \frac{1}{i}(Ib, b) = \frac{1}{i}(Ia, a),$$

from which our statement follows.

We will now assume that ρ is an eigenvalue of the matrix X_1 of multiplicity greater than one, $|\rho| = 1$, where $\mathscr{L}_\rho$ is the corresponding subspace of roots and $a_1, a_2, \ldots, a_m$ is a basis in the subspace. Then it follows from (2.1) that $\mathscr{L}'_\rho = U\mathscr{L}$ is the root subspace for the eigenvalue ρ of the matrix X_2. If we take as the basis in $\mathscr{L}'_\rho$ the vectors $b_j = Ua_j$, $j = 1, 2, \ldots, m$) we obtain the result that the Gramm matrices of the quadratic form $(1/i)(Ix,x)$ coincide in the subspaces $\mathscr{L}_\rho$ and $\mathscr{L}'_\rho$:

$$\left\| \frac{1}{i}(Ia_j, a_h) \right\| = \left\| \frac{1}{i}(Ib_j, b_h) \right\| \quad (j, h = 1, 2, \ldots, m).$$

Thus, these matrices have the same number of positive and negative /331
eigenvalues. This means that the same number of eigenvalues of the first and second kind of the matrices X_1 and X_2 coincided at the point ρ.

We prove that for arbitrary $X_1, X_2 \in \mathfrak{G}$ (2.1) implies $\zeta(X_1) = \zeta(X_2)$.

Let again X_1 and X_2 be given matrices. Since $\mathfrak{G}$ is a connected set, there exists a curve $U(t) \in \mathfrak{G}$, $0 \leqslant t \leqslant 1$, which connects the matrix U with the unit matrix E. Then, as the first relation in (2.1) implies, the curve $X(t) = U(t)^{-1}X_1U(t)$ will connect the matrices X_1 and X_2 in the group $\mathfrak{G}$. Since the relations

$$X(t) = U(t)^{-1}X_1U(t), \quad U(t)^{\tau}IU(t) = I,$$

are satisfied, which are analogous to the relations (2.1), we have, in accordance with what was proved above, the result that the eigenvalues together with their kind, coincide for the matrices X_1 and $X(t)$:

$$\zeta[X(t)] = \zeta(X_1) = \zeta_0.$$

2) At least one of the matrices X_1 and X_2 is not reduced to diagonal form. In this case, the statement of the theorem follows immediately from the fact that any matrix $X \in \mathfrak{G}$ can be connected by a continuous curve which remains the set $\zeta(X) = \zeta_0$ with a matrix which is reduced to diagonal form. In fact, to do this in the canonical decomposition given by Theorem 1.2, in all matrices [11] $\|Q_\varepsilon(\alpha), \overline{Q_\varepsilon(\varphi)}, \dot{Q}_\varepsilon(\lambda_0), \tilde{Q}_\varepsilon(0)\|$ ε must be connected continuously with zero. This proves Theorem 2.2.

An important point in Theorem 2.2 is that $\zeta(X)$ denotes the set of eigenvalues taking into account their kind. Thus, if X_1 and X_2 have the same eigenvalues (or even the same Jordan form), then generally X_1 and X_2 cannot be connected in the group $\mathfrak{G}$ without shifting the eigenvalues. Let us consider, for example, the similar matrices of order two $X_1 = e^{I\phi}$ and $X_2 = e^{-I\phi}$, $0 < \phi < \pi$. Let $X(t)$ be a continuous curve with the same eigenvalues, i.e. $X(t) = R(t)e^{I\phi}R(t)^{-1}$, such that $X(0) = e^{I\phi}$, $X(1) = e^{-I\phi}$. Since $R(t)$ is a matrix whose columns are the real and imaginary parts of the eigenvector of the matrix $X(t)$, we can choose an $R(t)$ which depends continuously on t. In addition to this, clearly $\det R(t) \neq 0$. For $t = 0$, we have: $e^{I\phi} = R(0)e^{I\phi}R(0)^{-1}$, and since $e^{I\phi} = E\cos\phi + I\sin\phi$, $R(0)I = IR(0)$, from which we easily obtain $R(0) > 0$. Hence, $\det R(1) > 0$. For $t = 1$, $e^{-I\phi} = R(1)e^{I\phi}R(1)^{-1}$, $R(1)I + IR(1) = 0$, which implies that $\det R(1) < 0$. The contradiction which was obtained shows that the matrices $e^{I\phi}$ and $e^{-I\phi}$ cannot be connected by a curve of the type described above. However, this does not contradict Theorem 2.2, since the eigenvalues $e^{i\phi}$ of the first kind /332 of the matrix $e^{I\phi}$ and $e^{-i\phi}$ of the matrix $e^{-I\phi}$ do not coincide,

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[11] We note that only at this point we used Theorem 1.2. The proof of the first proposition was based on the decomposition (1.2) which can be easily proved directly.

i.e. $\zeta(e^{i\varphi}) \neq \zeta(e^{-i\varphi})$.

The main purpose of this section is to prove that not only the complete image of a point but also the complete image of any region (of an open connected set) is a connected set. First, we introduce the following definition.

Definition. The mapping $x' = \zeta(x)$ of the topological space $R = \{x\}$ onto the topological space $R' = \{x'\}$ is called weakly open at the point x_0' if a point $x_0 \in R$, $\zeta(x_0) = x_0'$ can be found such that for any neighborhood O_{x_0} of the point there exists a neighborhood $O_{x_0'}$ of the point x_0' such that $O_{x_0'} \subset \zeta(O_{x_0})$.

A mapping which is weakly open at each point $x_0' \in R'$ is said to be simply weakly open.

This definition is an extension of the usual definition of an open mapping (the mapping $x' = \zeta(x)$ is open if for any point $x_0 \in R$ and its neighborhood O_{x_0} there exists a neighborhood $O_{x_0'}$ of the point $x_0' = \zeta(x_0)$ such that $O_{x_0'} \subset \zeta(O_{x_0})$). An open mapping is of course weakly open. The following example will show that the converse proposition is not true.

Let $R = \{\xi, \eta\}$ be a plane, R' the set of points which lie on the coordinate axes $\xi = 0$ and $\eta = 0$ in R, with real neighborhoods which are defined. We plot in the plane R the set of curves $\xi^2 - \eta^2 = a$, $-\infty < a < \infty$. To each point $x_0 = (\xi_0; \eta_0)$ in R, we make correspond the point $x_0' \in R' \subset R$, which is obtained when the line $\xi^2 - \eta^2 = \xi_0^2 - \eta_0^2 \in R'$ is intersected. It is easily seen that for any point x_0 which lies on the bisectrices of the coordinate angles which is different from the coordinate origin, for a "sufficiently small" neighborhood O_{x_0}, a neighborhood $O_{x_0'}$ does not exist such that $O_{x_0'} \subset \zeta(O_{x_0})$. Thus, this mapping is not open. However, as can be easily seen, it is weakly open (we must take $x_0 = x_0'$).

Lemma 1. Let $R = \{x\}$ be a locally connected topological space[12]

[12] We recall the definition. A locally connected topological space is a space in which, for every point x_0 and the neighborhood O_0, there exists a neighborhood $O \subset O_0$ which is a connected set. Connectedness is defined in the same sense as before.

$R' = \{x'\}$ be a topological space and $x' = \zeta(x)$ be a continuous, weakly open mapping of R onto R' such that the complete image of any point $x'_0 \in R'$ is a connected set in R. Then the complete image of any domain in R' will be a domain in R.

Proof. Let G' be a domain in R' and G be the complete image of G'. G is an open set, so that $\zeta(x)$ is a continuous mapping. Suppose that G is not connected. Suppose that the points x_1 and x_2 cannot be connected by a continuous curve in G. We denote by G_1 the connecting component of the set G containing x_1, i.e. the set of all $x \in G$, which can be connected with x_1 by curves which lie in G. /333
Since R is locally connected, G_1 is an open set, i.e. G_1 is a domain. Let $G_2 = G \setminus G_1$. Then because it is locally connected, the set G_2 is also open.

We introduce the notation

$$x'_1 = \zeta(x_1), \quad x'_2 = \zeta(x_2), \quad G'_1 = \zeta(G_1), \quad G'_2 = \zeta(G_2).$$

We will show that the intersection $G'_1 \cap G'_2$ is empty. Let us assume that the proposition is not true. Then there exist $y_1 \in G_1$, $y_2 \in G_2$, such that $\zeta(y_1) = \zeta(y_2) = y' \in G'_1 \cap G'_2$. By definition the set of all y in R such that $\zeta(y) = y'$ is connected. Therefore, y_1 and y_2 can be connected by a continuous curve in this set which is contained completely in G. But then the point y_2 belongs to G_1. Hence, the contradiction shows that $G'_1 \cap G'_2$ is empty.[13] It follows that G_j is the complete inverse image of G'_j ($j = 1, 2$). Since the sets G_j are open and the image $x' = \zeta(x)$ is weakly open, the G'_j ($j = 1, 2$) are also open. We will show that G' is not connected.

We select arbitrary $z'_1 \in G'_1$, $z'_2 \in G'_2$ and we assume that $z' = f(t)$, $0 \leqslant t \leqslant 1$ is a continuous curve connecting the points z'_1 and z'_2 in G'. We denote by A the class of all $t \in [0,1]$, such that for all $t' < t$ $f(t') \in G'_1$; $B = [0, 1] \setminus A$. The sets A and B are not empty. In fact, since $f(0) \in G'_1$, $f(1) \in G'_2$, G'_1 и G'_2, G'_1 and G'_2 are open and $f(t)$ is a

[13] We already obtained the result that G' is not connected in terms of another definition.

continuous function, all t sufficiently close to zero belong to the set A, and all t sufficiently close to one belong to the set B. If $\{t_1 \in A, t_2 \in B\}$, then $t_1 < t_2$. Therefore, (A, B) is a Dedekind cut. Let t_0 be the corresponding number $0 < t_0 < 1$. By definition of t_0 in any neighborhood of the point [illegible] = $f(t_0)$, there are points belonging to G_1' and G_2'. Since G_1' and G_2' are open, and do not intersect, z_0' belongs either to G_1' or to G_2' which is not possible. Hence, G' is not connected.

Thus, assuming that G is not connected, we obtained the result that G' is not connected. Consequently, G is connected, which proves Lemma 1.

Of course, Lemma 1 remains valid if we require that [illegible] = $\zeta(x)$ be an open mapping. However the mapping $\zeta = \zeta(X)$ under consideration is not open at the points $X \in \tilde{\Gamma}$, which correspond to the canonical matrices for the eigenvalues which are equal to one in absolute value. (At all other points, it can be shown, that this mapping is open.) This is easily shown for the case k = 1.

Let

$$X_0 = \begin{pmatrix} 1 & \varepsilon \\ 0 & 1 \end{pmatrix}, \quad \varepsilon \neq 0, \quad \zeta(X_0) = \zeta_0 = \{1, 1\}.$$

The matrix X_0 "lies" on the cone Γ^+ with center at the point E in the torus $\mathfrak{G}$ (see Fig. 1). Consequently, either "unstable" or "stable" matrices are close to it for which the multipliers are arranged in a certain fashion, for example, for which the multiplier of the first kind lies on the upper semicircle (depending on the sign of ε). But $\zeta = \{e^{i\varphi}; e^{-i\varphi}\}$ can be close to the point $\zeta_0 \in \Sigma$ both for $\phi > 0$ (multiplier of the first kind on the upper semicircle), and for $\phi < 0$ (multiplier of the first kind on the lower semicircle). Thus, at the point X_0 the mapping $\zeta = \zeta(X)$ is not open. /334

We note that the statement of Lemma 1 is not valid if we disregard the requirement that the mapping $x' = \zeta(x)$ be weakly open. Thus, for example, let R be the square abcd without the side cd, and let R' be the circle. Projecting all points of the square onto [a,d) and "convoluting" the half-open interval [a,d) into the circle, we obtain a continuous mapping $x' = \zeta(x)$ (which is not weakly open) such that the complete image of any point $x_0' \in R'$ is a connected set. However, the statement of Lemma 1 is not valid.

Theorem 2.3. The mapping $\zeta = \zeta(X)$ is weakly open.

Proof. We will consider an arbitrary $\zeta_0 \in \Sigma$. We will first assume that ζ_0 does not have repeated multipliers on the unit circle and on the real axis. Suppose

$$\zeta_0 = \{re^{i\psi}, re^{-i\psi}, \mu, e^{i\varphi_1}, e^{i\varphi_2}, \ldots ; r^{-1}e^{-i\psi}, \ldots\} \tag{2.2}$$

(first we write out the multipliers of the first kind). To the element ζ_0 we make correspond the canonical decomposition

$$T^{-1}X_0T = \begin{pmatrix} re^{I_2\psi} & 0 \\ 0 & r^{-1}e^{I_2\psi} \end{pmatrix} \oplus \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \oplus e^{I_2\varphi_1} \oplus e^{I_2\varphi_2} \oplus \ldots \tag{2.3}$$

For the matrix T, we take any matrix such that

$$T^{\tau}I^{-1}T = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \oplus I_2^{-1} \oplus I_2^{-1} \oplus I_2^{-1} \oplus \ldots \tag{2.4}$$

Then the matrix X_0 will be simplectic with eigenvalues equal to the multipliers of ζ_0. The $e^{i\phi_1}$, $e^{i\phi_2}$, ... will be the multipliers of the first kind for the matrix X_0. Thus, $\zeta(X_0) = \zeta_0$.

All $\zeta \in \Sigma$ which are sufficiently close to ζ_0 have the form

$$\zeta = \{r'e^{i\psi'}, r'e^{-i\psi'}, \mu', e^{i\varphi_1'}, e^{i\varphi_2'}, \ldots ; \ldots\}. \tag{2.5}$$

To each ζ we make correspond the matrix X using the formula

$$T^{-1}XT = \begin{pmatrix} r'e^{I_2\psi'} & 0 \\ 0 & (r')^{-1}e^{I_2\psi'} \end{pmatrix} \oplus \begin{pmatrix} \mu' & 0 \\ 0 & (\mu')^{-1} \end{pmatrix} \oplus e^{I_2\varphi_1'} \oplus e^{I_2\varphi_2'} \oplus \ldots \tag{2.6}$$

By virtue of (2.4) and (2.6), $X \in G$ and $\zeta(X) = \zeta$. The matrix X will be arbitrarily close to X_0, if the numbers ϕ', r', μ', ϕ_1' ... are sufficiently close to the numbers ϕ, r, μ, ϕ_1, ... Hence, for any neighborhood O_{X_0} we can find a neighborhood O_{ζ_0}, such that /335

$O_{\zeta_0} \subseteq \zeta(O_{X_0})$.

We will now assume that ζ_0 has repeated multipliers on the unit circle or on the real axis. The matrix $X_0 \in \mathfrak{G}$ will again be defined by formulas (2.3), (2.4). However, now ζ in any arbitrarily small neighborhood of ζ_0 can have a form which is different from (2.5).

We will consider three cases:

1) for ζ_0 the multipliers coincide at the points $\rho_0 = e^{\pm i\varphi_0} \neq \pm 1$.
2) $\rho_0 = \pm 1$
3) the multipliers coincide at the points $\rho_0 = \mu, \mu^{-1}$ on the real axis.

Of course, it is possible that for ζ_0 different cases can occur for different multipliers; and several multipliers may exist for which the same case occurs.

Suppose that m_1 multipliers of the first kind and m_2 multipliers of the second kind coincided at the point $\rho_0 = e^{i\phi_0}$.

In the first case, to each fourtuple of multipliers (two of the first kind and two of the second kind) for ζ_0

$$\zeta_0 = \{ \ldots, e^{i\varphi_0}, e^{-i\varphi_0}, \ldots ; \ldots, e^{-i\varphi_0}, e^{i\varphi_0}, \ldots \},$$

may correspond a fourtuple of multipliers for ζ which do not lie on the unit circle:

$$\zeta = \{ \ldots, re^{i\varphi}, re^{-i\varphi}, \ldots ; r^{-1}e^{-i\varphi}, r^{-1}e^{i\varphi}, \ldots \}. \tag{2.7}$$

In the second case, in addition to this, to each pair $\rho_0^+ = 1$, $\rho_0^- = 1$ of multipliers of the first and second kind for the element ζ_0

$$\zeta_0 = \{ \ldots, 1 \ldots ; \ldots, 1, \ldots \}.$$

may correspond the pairs μ, μ^{-1} for the element ζ,

$$\zeta = \{ \ldots, \mu, \ldots ; \ldots, \mu^{-1}, \ldots \}, \tag{2.8}$$

and similarly in the case $\rho_0^+ = \rho_0^- = -1$.

In the third case, the ζ which have the form

$$\zeta = \{ \ldots, re^{i\varphi}, re^{-i\varphi}, \ldots ; \ldots, r^{-1}e^{-i\varphi}, r^{-1}e^{i\varphi}, \ldots \}.$$

may be close to the element ζ_0

$$\zeta_0 = \{ \ldots, \mu, \mu, \ldots; \ldots, \mu^{-1}, \mu^{-1}, \ldots \},$$

To the multipliers ζ_0 and ζ which have the form (2.2) and (2.5), we make again correspond the factors in the right member of the decomposition (2.3) and (2.6). For these factors, the arguments do not change; therefore, we will not write them out below. In the second case, these arguments also do not change. The factors of the matrices X_0 and I^{-1} which correspond to the first case have the form:

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$$T^{-1}X_0T = e^{I_1\varphi_0} \oplus e^{I_1\varphi_0} \oplus \ldots = \begin{pmatrix} e^{I_2\varphi_0} & 0 \\ 0 & e^{I_1\varphi_0} \end{pmatrix} \oplus \ldots,$$
$$T^{\prime}I^{-1}T = I_2^{-1} \oplus -I_2^{-1} \oplus \ldots = \begin{pmatrix} I_2^{-1} & 0 \\ 0 & -I_2^{-1} \end{pmatrix} \oplus \ldots$$

It is clear from these decompositions that $X_0 \in \mathfrak{G}, \zeta(X_0) = \zeta_0$. We let $T = T_0S_0$ where $S_0 = S \oplus E_{2k-4}$, and S is an orthogonal matrix of order four which has the form

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} E_2 & E_2 \\ -I_2 & I_2 \end{pmatrix}.$$

We then obtain:

$$T_0^{-1}X_0T_0 = S \cdot \begin{pmatrix} e^{I_2\varphi_0} & 0 \\ 0 & e^{I_1\varphi_0} \end{pmatrix} \cdot S^{-1} \oplus \ldots = \begin{pmatrix} e^{I_2\varphi_0} & 0 \\ 0 & e^{I_1\varphi_0} \end{pmatrix} \oplus \ldots, \tag{2.9}$$

$$T_0^{\prime}I^{-1}T_0 = S \cdot \begin{pmatrix} I_2^{-1} & 0 \\ 0 & -I_2^{-1} \end{pmatrix} \cdot S^{\prime} \oplus \ldots = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \oplus \ldots \tag{2.10}$$

We make correspond to the element ζ (2.7) the matrix X, defined by the equation

$$T_0^{-1}XT_0 = \begin{pmatrix} re^{I_2\varphi} & 0 \\ 0 & r^{-1}e^{I_1\varphi} \end{pmatrix} \oplus \ldots \tag{2.11}$$

From a comparison of (2.10) and (2.11), it follows that $X \in \mathfrak{G}$.
Here, $\zeta(X) = \zeta$, and if ζ is sufficiently close to ζ_0, then r is close to 1, ϕ is close to ϕ_0 and the matrix X is close to X_0. Thus, for any neighborhood O_{X_0} a neighborhood O_{ζ_0} can be found such that $O_{\zeta_0} \subset \zeta(O_{X_0})$.

The matrix S can be defined as follows. The matrix must satisfy Eqs. (2.9) and (2.10). We seek S in the form

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Substituting this expression in (2.9), with $e^{I_2\varphi_0} = \cos\varphi_0 \cdot E_2 + \sin\varphi_0 \cdot I_2$, we obtain:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

which implies that all matrices A, B, C, D commute with I_2. A matrix which commutes with I_2 has the form $aE_2 + bI_2$ (incidentally, we can simply try to find A, B, C, D of this form). The set of matrices $\{aE_2 + bI_2\}$ forms a field which is isomorphic to the field of complex numbers $\{a + bi\}$. The complex conjugate corresponds to the transposed matrix. Making correspond the complex numbers α, β, γ, δ to the matrices A, B, C, D, we rewrite the remaining equation in (2.10) in the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This is equivalent to the equalities /337

$$|\alpha| = |\beta|, \quad |\gamma| = |\delta|, \quad \alpha\gamma^* - \beta\delta^* = i.$$

From the above, we easily obtain the general form of the matrix S which satisfies relations (2.9) and (2.10). In particular, we can take

$$\alpha = \beta = \frac{1}{\sqrt{2}}, \quad \gamma = -\frac{i}{\sqrt{2}}, \quad \delta = \frac{i}{\sqrt{2}}.$$

We then obtain a matrix S of the form shown above.

Finally, in the last, third case, we have:

$$T^{-1}X_0T=\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}\oplus\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}\oplus\dots,\quad T^{\tau}I^{-1}T=I_2^{-1}\oplus I_2^{-1}\oplus\dots, \tag{2.12}$$

$X_0\in\mathfrak{G},\ \zeta(X_0)=\zeta_0.$ As before, we let $T = T_0S_0$, where $S_0=S\oplus E_{2h-4}$ and S is the fourth order matrix

$$S=\left(\begin{array}{cc|cc}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ \hline 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{array}\right).$$

Then, from (2.12), we obtain:

$$T_0^{-1}X_0T_0=S\cdot\left(\begin{array}{cc|cc}\mu & 0 & 0 & 0\\ 0 & \mu^{-1} & 0 & 0\\ \hline 0 & 0 & \mu & 0\\ 0 & 0 & 0 & \mu^{-1}\end{array}\right)\cdot S^{-1}\oplus\dots=\begin{pmatrix}\mu E_2 & 0\\ 0 & \mu^{-1}E_2\end{pmatrix}\oplus\dots, \tag{2.13}$$

$$T_0^{\tau}I^{-1}T_0=S\cdot\begin{pmatrix}I_2^{-1} & 0\\ 0 & I_2^{-1}\end{pmatrix}\cdot S^{\tau}\oplus\dots=\begin{pmatrix}0 & E_2\\ -E_2 & 0\end{pmatrix}\oplus\dots. \tag{2.14}$$

To the element ζ we make correspond the matrix X

$$T_0^{-1}XT_0=\begin{pmatrix}re^{I_2\varphi} & 0\\ 0 & r^{-1}e^{I_2\varphi}\end{pmatrix}\oplus\dots. \tag{2.15}$$

Formulas (2.14) and (2.15) imply that $X\in\mathfrak{G}$, and from a comparison of (2.13) and (2.15) it follows that the matrix X is arbitrarily close to X_0 if the element ζ is sufficiently close to ζ_0. Thus, also in this case, for any neighborhood O_{X_0} there exists a neighborhood O_{ζ_0} such that $O_{\zeta_0}\subset\zeta(O_{X_0})$.

This proves Theorem 2.3.[14]

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[14]The proof is laborious because of the comparatively complex canonical structure of the matrices $X\in\mathfrak{G}$. Obviously, logically it would be simpler to prove first the analogous theorem for the algebra of the complex matrices K = iIH, H* = H, which are the matrices of symmetric transformations of pseudounitary spaces which have a simpler canonical structure, and then using the Cayley transformation pass onto the group $\mathfrak{G}^*$ of I-unitary complex matrices U, U*IU = I, and then by virtue of $\mathfrak{G}\subset\mathfrak{G}^*$ derive the theorem for the group $\mathfrak{G}$. However, this approach is just as long.

The following theorem which will be used below follows from /338
Theorems 2.1, 2.2, 2.3, and Lemma 1:

Theorem 2.4. For the mapping $\zeta = \zeta(X)$, the complete image of any region in Σ is a region in $\mathfrak{G}$.

3. Structure of the Group $\mathfrak{G}$

Theorem 2.4 enables us to reduce the study of the structure of the group $\mathfrak{G}$ from the point of view which interests us to a study of the set Σ. We will first consider the set $\tilde{\mathcal{O}}$.

Following [7], the elements ζ_1 and $\zeta_2 \in \tilde{\mathcal{O}}$ are said to be of the same type if when we move in the counterclockwise direction along the unit circle from the point $\rho = 1$, the multipliers of the first and second kind for ζ_1 and ζ_2 alternate in the same manner. I.e., ζ_1 and ζ_2 are of the same type if a continuous curve $\zeta(t)$ exists which connects ζ_1 and ζ_2 without intersecting $\hat{\Gamma}_0$. There are 2^k possible positions for the multipliers of the first and second kind on the unit circle. We will denote them by $\mu_1, \ldots, \mu_{2k}$. We will denote by $\hat{\mathcal{O}}^{(\mu)}$ the set $\zeta \in \Sigma$ for one type $\mu = \mu_j$. Clearly, $\hat{\mathcal{O}}^{(\mu)}$ is a domain. We denote by $\tilde{\mathcal{O}}^{(\mu)}$ the complete image $\tilde{\mathcal{H}}^{(\mu)}$ in the group $\mathfrak{G}$. By Theorem 2.4 $\tilde{\mathcal{O}}^{(\mu)}$ is also a domain.

Below we will need another property of the domains $\tilde{\mathcal{O}}^{(\mu)}$.

Definition. The set $\mathfrak{M} \subset \mathfrak{G}$ is said to be singly connected in $\mathfrak{G}$, if any closed curve which lies entirely in $\mathfrak{M}$ can be contracted into a point in the group $\mathfrak{G}$.

Clearly, a set which is singly connected in $\mathfrak{G}$ need not be connected ("into itself") (see, for example, the set $\mathfrak{M}_0$ in Fig. 3, below).

We will show that the domains $\tilde{\mathcal{O}}^{(\mu)}$ are singly connected in $\mathfrak{G}$. Let $X(t)$, $0 \leq t \leq 1$ be a closed curve in $\tilde{\mathcal{O}}^{(\mu)}$. We will number the eigenvalues of the matrix $X(t)$ on the upper semicircle in the order of increasing arguments. Since eigenvalues $\rho_j(t)$ of different kinds are not encountered as t varies, $0 \leq t \leq 1$, every point $\rho_j(t)$ moves as $0 \leqslant t \leqslant 1$ along the upper semicircle and $\rho_j(0) = \rho_j(1)$. Hence,

$$\operatorname{Arg} X(t)\big|_0^1 = \sum_{j=1}^{k} \Delta \arg \rho_j^+(t) = 0.$$

(the sum contains only the multipliers ρ_j^+ of the first kind). By Theorem 1.1, the curve X(t) can be contracted into a point.

We prove the following theorem:

Theorem 3.1. The set $\tilde{\mathfrak{S}} \subset \mathfrak{S}$ decomposes into 2^k nonintersecting singly connected domains $\tilde{\mathfrak{S}}^{(\mu)}$ in the group $\mathfrak{S}$, each of which is characterized by a certain distribution of the multipliers of the first and second kind on the unit circle.

This theorem was proved in the work of I.M. Gel'fand and V.B. Lidskiy ([7], Par. 6 and Par. 8, Lemma 4) by constructing actual /339 curves. For stability regions, this approach is undoubtedly simpler, since in stability regions all matrices are reduced to diagonal form. When the instability regions are studied, such curves can not be constructed, and moreover their deformations can not be studied because along such curves the canonical structure of the matrix can change in a very complex way (see Theorem 1.2).

We will now consider the set $\tilde{\mathscr{H}}$. When k = 1, the set $\mathscr{H} = \zeta(\tilde{\mathscr{H}})$ decomposes into two domains $\mathscr{H}^{(\nu_1)}$ and $\mathscr{H}^{(\nu_2)}$, which correspond to two possible unstable types ν_1 and ν_2. By Theorem 2.4, two domains and $\tilde{\mathscr{H}}^{(\nu_2)}$ into which $\tilde{\mathscr{H}}$ decomposes correspond to these (see also Fig. 1).

Theorem 3.2. For k > 1, the set $\tilde{\mathscr{H}}$ is connected.

Proof. Since the set $\mathscr{H} = \zeta(\tilde{\mathscr{H}})$ is clearly open, by Theorem 2.4 it suffices if we prove that $\mathscr{H}$ is connected.

Let $\zeta_0 = \left\{\frac{1}{2};\ \frac{1}{2}, \ldots, \frac{1}{2};\ 2, 2, \ldots, 2\right\} \in \mathscr{H}$. We connect an arbitrary point $\zeta \in \mathscr{H}$ with ζ_0 by a continuous curve which lies entirely in $\mathscr{H}$.

We first assume that ζ has multipliers on the positive real axis. Then these can be brought to the points $\rho = 1/2$ and $\rho = 2$.

After this, all remaining multipliers may also be brought to these points. In the process, we do not leave $\mathscr{H}$, since during the deformation some of the multipliers always remain in the interior of the unit circle. If for ζ there are no multipliers on the positive real axis, but there is at least one fourtuple of multipliers which do not lie on the unit circle and on the negative real axis, these fourtuples can be shifted continuously onto the real positive axis while staying in $\mathscr{H}$, and the point which is obtained, as shown above, can be connected with ζ_0. All we must analyze now is the case when for ζ all multipliers which are different in absolute value from one lie on the negative real axis. If their number is greater than four, the fourtuple of multipliers can be displaced into the region $|\rho| \neq 1, \mathfrak{J}\rho \neq 0,$ and we have the previous case. Finally, if there are only two multipliers, the remaining multipliers lie on the unit circle ($k > 1$!). A pair of these multipliers (of the first and second kind) can be brought to the real negative axis and the fourtuple which is obtained can be shifted to the region $|\rho| \neq 1, \mathfrak{J}\rho \neq 0$. Thus, we again have the preceding case.

Thus, in either case ζ can be connected with ζ_0 in the region $\mathscr{H}$, i.e. $\tilde{\mathscr{H}}$ is connected which was to be proved.

The connectedness of the set $\tilde{\mathscr{H}}$ when $k > 1$ "is not convenient" in a certain sense. In the next paragraph, we will show that the connectedness of the set $\tilde{\mathscr{H}}$ implies the connectedness of the set $\mathscr{H} \subset \mathfrak{H}$ (when $k > 1$). At the same time, when $k = 1$, from the fact that $\tilde{\mathscr{H}}$ decomposes into two domains $\tilde{\mathscr{H}}^{(\nu_1)}$ and $\tilde{\mathscr{H}}^{(\nu_2)}$, it follows that $\mathscr{H}$ decomposes into a countable number of domains $\mathscr{H}_n^{(\nu)}$ $(\nu = \nu_1, \nu_2;$ /340
$n = 0, \pm 1, \pm 2, \ldots)$ For each of these domains, the following theorem [12] is valid:

If $H_1(t) \leqslant H(t) \leqslant H_2(t)$ and the matrices $H_1(t)$ and $H_2(t)$ belong to the same instability region $\mathscr{H}_n^{(\nu)}$ $(\nu = \nu_1, \nu_2; n = 0, \pm 1, \pm 2, \ldots)$, then the matrix $H(t)$ belongs to the same instability region.

From this theorem, various efficient instability criteria can be derived. What is the analogue of the theorem for the unstable case when $k > 1$? The proposition which suggests itself, namely that the theorem holds for the entire set $\mathcal{H}$, is immediately refuted by simple examples. Nevertheless, it turns out that the set $\mathcal{H}$ can be broken up by some "surface" Γ into a countable number of domains in such a way that for some of these domains the above theorem holds. The set Γ is the set of those systems (0.1) whose monodromy matrices have repeated multipliers of different kinds (see table on p. 10).

We will consider ζ_1 and $\zeta_2 \in \hat{\mathcal{H}} \setminus \hat{\Gamma}$. We will say that ζ_1 and ζ_2 have the same multiplier distribution (or simply the same type ν), if: 1) they have the same type of multiplier distribution on the unit circle in the sense defined earlier, 2) the multiplier pairs on the real positive axis for ζ_1 and ζ_2 have the same parity, 3) and if the same holds also for the negative real axis.[15]

We will say that X_1 and $X_2 \in \tilde{\mathcal{H}} \setminus \tilde{\Gamma}$ have the same type of distribution of eigenvalues if $\zeta(X_1)$ and $\zeta(X_2)$ are of the same type.

Theorem 3.3. The set $\tilde{\mathcal{H}} \setminus \tilde{\Gamma}$ is the union of $N = 2(2^k - 1)$ simply connected non-intersecting domains $\tilde{\mathcal{H}}^{(\nu)}$ $(\nu = \nu_1, \ldots, \nu_N)$ in $\mathfrak{S}$, each of which is the set of all matrices $X \in \tilde{\mathcal{H}} \setminus \tilde{\Gamma}$ having a distribution of eigenvalues of the same type.

Proof. If we denote by $\hat{\mathcal{H}}^{(\nu)}$ the set $\{\zeta\}$ of a given type ν, then

$$\hat{\mathcal{H}} \setminus \hat{\Gamma} = \bigcup_{\nu} \hat{\mathcal{H}}^{(\nu)}.$$

For continuous deformations ζ, for which the multipliers of different kinds do not coincide, only fourtuples of multipliers can either leave or be brought to the real axis, but for such deformations the type ζ does not change. On the other hand, it is clear that elements ζ of the same type can be deformed by such deformations into one another. Thus, the sets $\hat{\mathcal{H}}^{(\nu)}$ are non-intersecting domains. By Theorem 2.4, their images, the sets $\tilde{\mathcal{H}}^{(\nu)} \subset \mathfrak{S}$ are also non-intersecting domains.

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[15] It is easily seen that condition 3) follows from 1) and 2).

We will show that these domains are simply connected in $\mathfrak{S}$. /341
Let $X(t)$, $0 \leqslant t \leqslant 1$ be a closed curve in $\tilde{\mathcal{H}}^{(\nu)}$. Along any continuous curve (not necessarily closed) for multipliers of the first kind, outside the unit circle

$$\sum_j \Delta \operatorname{Arg} \rho_j = 0, \tag{3.1}$$

since these multipliers are distributed symmetrically relative to the real axis. Along a closed curve for multipliers of the first kind on the unit circle equality (3.1) also holds: This is proved in the same way as for the domains $\tilde{\mathcal{O}}^{(\mu)}$. Thus, $\operatorname{Arg} X(t)|_0^1 = 0.$ By Theorem 1.1, the curve $X(t)$ can be contracted into a point, i.e. the domains $\tilde{\mathcal{H}}^{(\nu)}$ are simply connected in $\mathfrak{S}$.

We will determine the number of different types ν. All types ν will be broken up into two classes: 1) the types ν for which the elements ζ have an even number of multipliers on the interval $(-1, 0)$, 2), and types ν for which the corresponding number is odd. We will deform ζ by displacing the fourtuples of multipliers without encountering multipliers of different kinds from the region $|\rho| \neq 1$ to the positive real axis. For the elements ζ of the first class after such deformation, there will be $2j$ multipliers on the unit circle, and the remaining multipliers will lie on the real positive axis $0 \leqslant j < k$. For the elements ζ of the second class, there will be two multipliers on the negative real axis, $2j$ on the unit circle, and the remaining multipliers will lie on the negative real axis $0 \leqslant j \leqslant k$. Since on the unit circle the $2j$ multipliers of the first and second kind can be distributed in 2^j ways, the total number of "unstable" types ν is equal to

$$N = 2\sum_{j=0}^{k-1} 2^j = 2(2^k - 1).$$

This proves Theorem 3.3.

Theorem 3.4. The sets $\tilde{M}_\alpha$ and $\tilde{m}_\alpha$ are domains for $k > 1$; for $k = 1$, $\tilde{m}_\alpha$ is a domain, and $\tilde{M}_\alpha$ decomposes into two non-intersecting domains:

$$\tilde{M}_\alpha = \tilde{M}_\alpha^{(+)} \cup \tilde{M}_\alpha^{(-)}.$$

In fact, repeating the proof of Theorem 3.2, we obtain the result that the corresponding statement is also valid for the sets $\hat{M}_\alpha, \hat{m}_\alpha$. When k = 1, $\hat{M}_\alpha^{(+)}$ is the set of elements $\zeta = \{\mu; \mu^{-1}\}, 0 < \mu < e^{-\alpha}$; for $\tilde{M}_\alpha^{(-)}$ $0 < -\mu < e^{-\alpha}$.

Theorem 3.4 follows from Theorem 2.4.

Lemma. Let $\mathfrak{M}$ be a domain in the group $\mathfrak{G}$. The arbitrary matrices X_1 and X_2 from $\mathfrak{M}$ can be connected by a continuous curve $X(t)$, $0 \leqslant t \leqslant 1$ which lies entirely in $\mathfrak{M}$, and which has the property that dX/dt exists everywhere except possibly at a finite number of points $t_0 = 0 < t_1 < \ldots < t_{m-1} < t_m = 1$, and the derivative dX/dt is continuous on the intervals (t_j, t_{j+1}). (From here on, we will call such curves piecewise smooth.) /342

It follows from this lemma that any two matrices from $\mathfrak{G}$ can be connected by a piecewise smooth curve.

Proof. Let us connect X_1 and X_2 by a continuous curve $X_0(t)$, $X_0(0) = X_1, X_0(1) = X_2$. We will encircle each point $X_0(t)$ by a neighborhood $\tilde{O}_t$ such that: 1) $\tilde{O}_t \subset \mathfrak{M}$, 2) for any two matrices $X', X'' \in \tilde{O}_t$ the matrix $X' \cdot (X'')^{-1}$ has no eigenvalues equal to -1.[16] From the set of coverings $\tilde{O}_t$ we select a finite covering $\tilde{O}_{t_j} = \tilde{O}_j$ (j = 1, 2, ..., q). Discarding the unnecessary neighborhoods, we will assume that the intersections $\tilde{O}_j \cap \tilde{O}_{j+1}$ are not empty. We select an arbitrary point $X'_j \in \tilde{O}_j \cap \tilde{O}_{j+1}$ $(j = 1, 2, \ldots, q-1)$. We set $X'_0 = X_1$, $X'_q = X_2$. We will prove our statement if in the neighborhood $\tilde{O}_j$ we connect the points X'_{j-1} and X'_j by a continuously differentiable curve.

The transformation

$$Y = X \cdot (X'_j)^{-1}$$

will map the neighborhood $\tilde{O}_j$ into some neighborhood $\tilde{O}$ of the matrix $E \in \mathfrak{G}$, and the matrices X'_j and X'_{j+1} into the matrices E and $Y_0 = X'_{j+1} \cdot (X'_j)^{-1}$, where any matrix Y in this neighborhood

[16] This can be done since for a sufficiently small neighborhood $\tilde{O}_t$ the matrix $X' \cdot (X'')^{-1}$ is arbitrarily close to the unit matrix.

has no eigenvalues which are equal to -1. Therefore, for all matrices in this neighborhood the Cayley transformation

$$C = I\frac{Y-E}{Y+E}, \quad Y = \frac{E-IC}{E+IC}. \tag{3.2}$$

is defined. Here C is a real symmetric matrix, such that det(E+IC) ≠ ≠ 0. Conversely, any such matrix C defines in the second equation in (3.2) the matrix $Y \in \mathfrak{S}$. The neighborhood $\tilde{O}$ will be mapped into some neighborhood O of the point C = 0 of $\frac{k(k+1)}{2}$ -dimensional Euclidian space of symmetric matrices. The points Y = E and Y_0 will be mapped into C = 0 and C_0. Connecting the matrix C_0 with the matrix C = 0 by a smooth curve which lies in O, and making the inverse transformations, we obtain the required curve X(t).

Theorem 3.5. Let $X_1(t)$ and $X_2(t)$, $0 \leqslant t \leqslant 1$ be piecewise smooth curves and let U(t, s), $0 \leqslant t,s \leqslant 1$ be a deformation of one curve into the other

$$U(t,0) = X_1(t),\ U(t,1) = X_2(t)$$

U(t,s) is a continuous function over the set t, s for $0 \leqslant t,s \leqslant 1$. Then a deformation V(t,s), $0 \leqslant t,s \leqslant 1$ of the curve $X_1(t)$ into the curve $X_2(t)$ exists such that:

1) V(t,s) is a differentiable function of t and s, $0 \leqslant t,s \leqslant 1$, /343 except possibly for a finite number of values of t and a finite number of values of s; when s is fixed, V(t,s) is a piecewise smooth curve.

2) The curves V(0,s) and U(0,s) and also V(1,s) and U(1,s) are homotopic. If U(0,s) = U_0 = const, then V(0,s) = U_0, if U(1,s) = U_1 = const, then also V(1,s) = U_1.

Proof. Let

$$\varepsilon_0 = \inf \| X - Y \|$$

over all X, $Y \in \mathfrak{S}$, satisfying the condition

$$\det(X + Y) = 0$$

Since for $X \in \mathfrak{G}$ det X = 1, clearly $\varepsilon_0 > 0$. We select an $\varepsilon_0 > 0$ such that for

$$|t-t'|+|s-s'| \leqslant \delta$$
$$\|U(t', s')-U(t, s)\| < \varepsilon_0.$$

We break up the square $0 \leqslant s,t \leqslant 1$ by the lines $t = t_j$, $s = s_k$ parallel to the axes t and s into a number of rectangles whose sides are smaller than δ. For

$$t_j \leqslant t \leqslant t_{j+1}, \quad s_k \leqslant s \leqslant s_{k+1} \tag{3.3}$$

U(t,s) belongs to the neighborhood which with the aid of (3.2) is mapped analytically onto the neighborhood O of the point C = E. We replace as shown above, the curve $U(t,s_k)$ by a piecewise smooth curve $V(t,s_k)$, and we can assume that $\frac{dV(t, s_k)}{dt}$ exists and is continuous when $t \neq t_j$. Then, in each rectangle (3.3) we interpolate the corresponding C(t,s) linearly. The matrix of functions V(t,s) which is obtained satisfies conditions 1) and 2), which was to be proved.

Because of this theorem, below we can restrict ourselves only to considering piecewise smooth curves.

4. Structure of the Set $\mathfrak{H} = \{H(t)\}$

To each matrix H(t) of the coefficients of system (0.1) corresponds the monodromy matrix $X = \Phi[H(t)] \in \mathfrak{G}$ of system (0.1). The mapping X = Φ[H(t)] is clearly a continuous mapping of $\mathfrak{H}$ onto $\mathfrak{G}$ (see, for example, [12], pp. 38-39).

As we have shown in Par. 1, the "point" $H(t) \in \mathfrak{H}$ can be identified with the piecewise smooth curve $X(t) \in \mathfrak{G}$, which begins in E and ends in the point X = Φ[H(t)].

Below, when we discuss curves, we will have in mind piecewise smooth curves.

We will adopt the following notation. The topological product of the paths $X_1(t)$ and $X_2(t)$, $0 \leqslant t \leqslant 1$, $X_1(1) = X_2(0)$ will be denoted by $X_1(t) \times X_2(t)$. Thus,

$$X_1(t) \times X_2(t) = X_3(t) = \begin{cases} X_1(2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ X_2(2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

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The paths traversed in the opposite direction will be denoted by $X(t)^{[-1]}$ as follows:

$$X(t)^{[-1]} = X(1-t), \quad 0 \leqslant t \leqslant 1.$$

A closed path $X(t)$ passed over m times will be denoted by $X(t)^{[m]}$.

All the operations above on piecewise smooth curves give again piecewise smooth curves.

We will call two curves $X_1(t)$ and $X_2(t)$, $0 \leqslant t \leqslant 1$ which begin at a common point and end in some set $\mathfrak{M} \subset \mathfrak{G}$, homotopic modulo $\mathfrak{M}$, and we will write

$$X_1(t) \sim X_2(t) \mod \mathfrak{M}, \tag{4.1}$$

if they can be deformed into one another without displacing the endpoint from $\mathfrak{M}$.

Homotopic curves will be denoted as follows:

$$X_1(t) \sim X_2(t)$$

A necessary and sufficient condition that $X_1(t)$ and $X_2(t)$ be homotopic is that $Y(t) \in \mathfrak{M}$, $0 \leqslant t \leqslant 1$, $Y(0) = X_1(1)$, $Y(1) = X_2(1)$ exist in mod $\mathfrak{M}$, such that the paths $X_1(t) \times Y(t)$ and $X_2(t)$ are homotopic.

In fact, if such curve exists, the deformation

$$Y(t, s) = \begin{cases} X_1[(s+1)t], & 0 \leqslant t \leqslant \frac{1}{s+1}, \\ Y[(s+1)t - s], & \frac{1}{s+1} \leqslant t \leqslant 1, \end{cases}$$

$0 \leqslant s \leqslant 1$ deforms the path $X_1(t)$ into the path $X_1(t) \times Y(t)$ which can be deformed into $X_2(t)$ without displacing the endpoints. During both deformations, the endpoint remains in $\mathfrak{M}$; hence, (4.1) holds.

Conversely, if (4.1) is satisfied and $X(t,s)$ is the corresponding deformation, then we can take for $Y(t)$, $Y(t) = X(1,t)$.

Theorem 4.1. Let $\tilde{\mathfrak{M}}$ be a domain in $\mathfrak{G}$ and $\mathfrak{M}$ be its complete image in $\mathfrak{H}$, $\Phi(\mathfrak{M}) = \tilde{\mathfrak{M}}$. If $\tilde{\mathfrak{M}}$ is a simply connected set in $\mathfrak{G}$, then $\mathfrak{M}$ decomposes into a countable number of homeomorphic domains $\mathfrak{M}_n$:

$$\mathfrak{M} = \bigcup_{-\infty}^{+\infty} \mathfrak{M}_n, \quad \mathfrak{M}_{n_1} \cap \mathfrak{M}_{n_2} = 0 \quad \text{when} \quad n_1 \neq n_2, \tag{4.2}$$

such that $\Phi(\mathfrak{M}_n) = \tilde{\mathfrak{M}}$.

Proof. We select an arbitrary point $X_0 \in \mathfrak{M}$ and we connect it /345 by the curve $X^{(0)}(t)$, $0 \leqslant t \leqslant 1$ with the matrix E. $\mathfrak{M}$ is the set of all curves which begin in E and end in $\tilde{\mathfrak{M}}$ in such a way that $X_0(t) \in \mathfrak{M}$.

The set of curves $X_0(t)$ which are homotopic modulo $\tilde{\mathfrak{M}}$ will be denoted by $\mathfrak{M}_0$, $\mathfrak{M}_0 \subset \mathfrak{M}$.

Let $U^{(n)}(t)$, $0 \leqslant t \leqslant 1$ be a curve which begins and ends in E such that

$$\operatorname{Arg} U^{(n)}(t)\big|_0^1 = 2\pi n.$$

(For $U^{(n)}(t)$ we can take the curve (1.8), Par. 1.)

The curve

$$X^{(n)}(t) = U^{(n)}(t) \times X^{(n)}(t) \tag{4.3}$$

belongs to $\mathfrak{M}$. The set of curves $X^{(n)}(t)$ which are homotopic modulo $\tilde{\mathfrak{M}}$ will be denoted by $\mathfrak{M}_n$, $\mathfrak{M}_n \subset \mathfrak{M}$. Clearly, the sets $\mathfrak{M}_n$ are connected and open, and $\Phi(\mathfrak{M}_n) = \tilde{\mathfrak{M}}$.

We will show that $\mathfrak{M} = \bigcup_{-\infty}^{+\infty} \mathfrak{M}_n$. Let $X(t) \in \mathfrak{M}$, $X(1) = X_1$.[17]

We connect X_1 and X_0 by the curve $Y(t) \in \tilde{\mathfrak{M}}$, $0 \leqslant t \leqslant 1$. The curve

$$X_2(t) = X_1(t) \times Y(t)$$

which consists of the curves $X_1(t)$ and $Y(t)$ ends in X_0. Hence,

$$\operatorname{Arg} X_2(t)\big|_0^1 - \operatorname{Arg} X^{(0)}(t)\big|_0^1 = 2\pi m$$

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[17] We note that when we write $X(t) \in \mathfrak{M}$, we mean that the curve X(t) belongs to the class of curves $\mathfrak{M}$. The notation $X(t) \in \tilde{\mathfrak{M}}$ means that for a fixed t the matrix $X(t) \in \tilde{\mathfrak{M}} \subset \mathfrak{G}$.

for some m = 0, ±1, ±2, ... Then,

$$\operatorname{Arg} X_2(t)\big|_0^1 - \operatorname{Arg} X^{(m)}(t)\big|_0^1 = 0,$$

i.e. $X_2(t) \sim X^{(m)}(t)$. Hence,

$$X_1(t) \sim X^{(m)}(t) \mod \mathfrak{M}$$

and $X_1(t) \in \mathfrak{M}_m$.

The domains $\mathfrak{M}_n$ do not intersect. In fact, let

$$X(t) \in \mathfrak{M}_{n_1} \cap \mathfrak{M}_{n_2}, \quad n_1 \neq n_2.$$

We then have

$$X(t) \sim X^{(n_1)}(t) \mod \tilde{\mathfrak{M}}, \quad X(t) \sim X^{(n_2)}(t) \mod \tilde{\mathfrak{M}}.$$

Hence, /346

$$X^{(n_1)}(t) \sim X^{(n_2)}(t) \mod \tilde{\mathfrak{M}}.$$

Then a curve $Y(t) \in \tilde{\mathfrak{M}}$ exists such that

$$X^{(n_1)}(t) \times Y(t) \sim X^{(n_2)}(t).$$

The curve Y(t) is closed and

$$\operatorname{Arg} Y(t)\big|_0^1 = \operatorname{Arg} X^{(n_2)}(t) - \operatorname{Arg} X^{(n_1)}(t) = 2(n_2 - n_1)\pi \neq 0.$$

Hence, the set $\tilde{\mathfrak{M}}$ is not simply connected in $\mathfrak{G}$. The contradiction which was obtained shows that the domains $\mathfrak{M}_{n_1}$ and $\mathfrak{M}_{n_2}$ do not intersect.

What we must still prove is that the domains $\mathfrak{M}_n$ are homeomorphic to one another. To each curve $X_0(t) \in \mathfrak{M}_0$ we make correspond the curve $X_n(t)$ in accordance with

$$X_n(t) = X_0(t) \cdot U^{(n)}(t) \tag{4.4}$$

Let

$$U^{(n)}(t, s) = \begin{cases} U^{(n)}(kt) & \text{for } 0 \leqslant t \leqslant \frac{1}{k}, \\ E & \text{for } \frac{1}{k} \leqslant t \leqslant 1, \end{cases}$$

$$X_0(t, s) = \begin{cases} E & \text{for } 0 \leqslant t \leqslant 1 - \frac{1}{k}, \\ X_0[k(t-1)+1] & \text{for } 1 - \frac{1}{k} \leqslant t \leqslant 1, \end{cases}$$

where $k = s + 1$, $0 \leqslant s \leqslant 1$. The deformation

$$X(t, s) = X_0(t, s) \cdot U^{(n)}(t, s), \quad 0 \leqslant s \leqslant 1,$$

deforms the curve (4.4)(s=0) into a curve of the form (4.3) ($X^{(0)}(t) = X_0(t)$) belonging to $\mathfrak{M}_n$ without displacing the endpoints. Hence, $X_n(t) \in \mathfrak{M}_n$ and formula (4.4) establish the homeomorphism of the domains $\mathfrak{M}_n$ and $\mathfrak{M}_0$. This proves Theorem 4.1.

Remark. The following rule for numbering the domains $\mathfrak{M}_n$ follows from the proof of Theorem 4.1. An arbitrary index can be assigned to a fixed domain $\mathfrak{M}_n$. Then the remaining domains $\mathfrak{M}_n$ can be numbered in such a way that for any curves $X_1(t) \in \mathfrak{M}_{n_1}$ and $X_2(t) \in \mathfrak{M}_{n_2}$ the relation

$$\operatorname{Arg} X_2(t)\big|_0^1 - \operatorname{Arg} X_1(t)\big|_0^1 = \operatorname{Arg} Y(t)\big|_0^1 + 2(n_2 - n_1)\pi, \tag{4.5}$$

will be satisfied, where $Y(t)$ is any curve which lies entirely in $\tilde{\mathfrak{M}}$, and which connects the points $X_1(1) = Y(0)$ and $X_2(1) = Y(1)$.

Let $\tilde{\mathfrak{M}}$ be a domain which is not simply connected in $\mathfrak{G}$. Then a closed curve $V(t)$, $0 \leqslant t \leqslant 1$ exists in it

$$\operatorname{Arg} V(t)\big|_0^1 = 2\pi m \neq 0.$$

Without loss of generality, we can assume $m > 0$. The smallest m will be called the index of the domain $\tilde{\mathfrak{M}}$. Domains which are simply connected in $\mathfrak{G}$ can be assigned an index which is equal to zero. /347

Theorem 4.2. The complete image $\mathfrak{M}$ of the domain $\tilde{\mathfrak{M}} \subset \mathfrak{G}$ with index $m > 0$ decomposes[18] in $\mathfrak{H}$ into m mutually homeomorphic domains $\mathfrak{M}_j$ ($j = 0, 1, \ldots, m-1$). In particular, if a closed trajectory $V(t)$, $0 \leqslant t \leqslant 1$ exists in $\tilde{\mathfrak{M}}$ such that $\operatorname{Arg} V(t)\big|_0^1 = 2\pi$,

[18] When we say that a set $\mathfrak{M}$ decomposes into a series of domains $\mathfrak{M}_j$, we mean that this set is a union of non-intersecting domains $\mathfrak{M}_j$.

$\mathfrak{M}$ is a domain.

Proof. Let $V(t)$ be the closed trajectory which exists by hypothesis, such that $\operatorname{Arg} V(t)|_0^1 = 2\pi m > 0, V(t) \in \tilde{\mathfrak{M}}$. We will connect the matrices E and V(0) by the curve $X^{(0)}(t)$, $X^{(0)}(0) = E$, $X^{(0)}(1) = V(0)$. Let $U^{(n)}(t)$ be the same curves which were used in proving Theorem 4.1, and define $X^{(n)}(t)$ by formula (4.3).

We will denote by $\mathfrak{M}_j$ $(j = 0, 1, \ldots, m-1)$ the set of trajectories which are homotopic to $X^{(j)}(t)$ modulo $\tilde{\mathfrak{M}}$. Clearly, $\mathfrak{M}_j$ are domains and $\Phi(\mathfrak{M}_j) = \tilde{\mathfrak{M}}$.

1) We will show that

$$\mathfrak{M} = \bigcup_{j=0}^{m-1} \mathfrak{M}_j. \tag{4.6}$$

Let $X(t) \in \mathfrak{M}$, $X(1) = X \in \tilde{\mathfrak{M}}$. We will connect X with V(0) by the curve $Y_0(t)$, $Y_0(0) = X$, $Y_0(1) = V(0)$, $Y_0(t) \in \tilde{\mathfrak{M}}$. Let

$$\operatorname{Arg} [X(t) \times Y_0(t) \times X^{(0)}(t)^{[-1]}]|_0^1 = 2\pi n.$$

Then

$$\operatorname{Arg} [X(t) \times Y_0(t) \times V(t)^{[k]} \times X^{(0)}(t)^{[-1]}]|_0^1 = 2\pi(n + km).$$

We choose k (positive or negative) in such a way that

$$0 \leqslant j = n + km < m.$$

Then

$$\operatorname{Arg} [X(t) \times Y_0(t) \times V(t)^{[k]}]|_0^1 = 2\pi j + \operatorname{Arg} X^{(0)}(t)|_0^1,$$

i.e.

$$\operatorname{Arg} [X(t) \times Y(t)]|_0^1 = \operatorname{Arg} X^{(j)}(t)|_0^1,$$

where $Y(t) = Y_0(t) \times V(t)^{[k]} \in \tilde{\mathfrak{M}}$. Thus, $X(t) \times Y(t) \sim X^{(j)}(t)$, $X(t) \in \mathfrak{M}_j$, which proves (4.6).

2) The domains $\mathfrak{M}_j$ do not intersect. Let

$$X(t) \in \mathfrak{M}_k \cap \mathfrak{M}_l, \quad 0 \leqslant k < l \leqslant m.$$

We have

$$X(t) \sim X^{(k)}(t) \mod \widetilde{\mathfrak{M}}, \quad X(t) \sim X^{(l)}(t) \mod \widetilde{\mathfrak{M}}.$$

Hence,

$$X^{(k)}(t) \sim X^{(l)}(t) \mod \widetilde{\mathfrak{M}}.$$

Then a curve $Y(t) \in \widetilde{\mathfrak{M}}$ exists such that

$$X^{(k)}(t) \times Y(t) \sim X^{(l)}(t).$$

The curve Y(t) is closed, since $X^{(k)}(1) = X^{(l)}(1)$ and

$$\operatorname{Arg} Y(t)\Big|_0^1 = 2(l-k)\pi < 2m\pi,$$

which contradicts the hypothesis that the index of $\mathfrak{M}$ is equal to m. This proves Theorem 4.2.

Thus, we solved the problem which was formulated at the end of the introduction. The index m of the domain is usually determined easily after the domain $\mathfrak{M}$ is given.

The connection between the statements of Theorems 4.1 and 4.2 and the concept of a space covering is easily established. The following can also be proved. Let Ω be the set of all paths which are homotopic to zero, which begin and end in E and let $\mathfrak{U}$ be a covering for the group $\mathfrak{G}$. The space $\mathfrak{H}$ is homeomorphic to the topological product $\Omega \times \mathfrak{U}$, and because of the correspondence which was established, the boundedness and unboundedness properties, the order, etc. of the solutions are only determined by the "projection" H(t) in $\mathfrak{U}$.

We recall that $\mathfrak{U}$ is a finite dimensional space of the same dimension as $\mathfrak{G}$, so that it can be studied in the same way as the model of the functional space $\mathfrak{H}$.

For k = 1, the statements of Theorems 4.1 and 4.2 are very clear. In this case, as we have shown above, the group $\mathfrak{G}$ is homeomorphic to the interior of a torus. The index m of the domain $\mathfrak{M}$ is a number which indicates how many times the domain $\mathfrak{M}$ "was twisted" in $\mathfrak{G}$. Thus, for example, the indices of the domains $\mathfrak{M}_0$ and $\mathfrak{M}_2$ shown in Fig. 3 are equal to zero and two,respectively. If we cut $\mathfrak{G}$ along some surface S (see Fig. 3) and take the countable number of pieces which were so obtained, and join these together, identifying

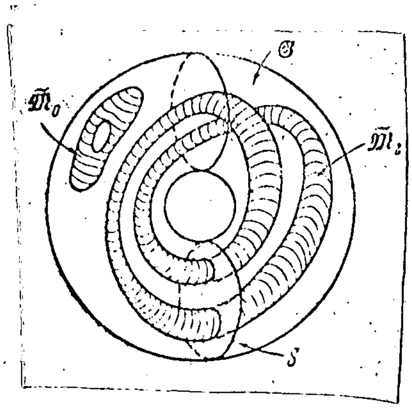

Fig. 3.

identical points, we obtain the "rope" $\mathfrak{A}$, which by virtue of the homeomorphism $\mathfrak{H} = \mathfrak{Q} \times \mathfrak{A}$, can be considered as the model of the space $\mathfrak{H}$. It is easily seen that in the process $\mathfrak{M}_0$ we will "generate" a countable number of domains, and $\mathfrak{M}_2$ will "generate" two domains in $\mathfrak{H}$. This is also stated by Theorems 4.1 and 4.2.

Theorem 4.3. 1^0. The set m_α for any k and the set $\mathcal{H}$, M_α /349 for $k \geq 1$ are domains.

When k = 1, the sets $\mathcal{H}$ and M_α decompose into two series of domains

$$\mathcal{H}_n^+,\ \mathcal{H}_n^-,\ M_{\alpha,n}^+,\ M_{\alpha,n}^- \quad (n = 0,\ \pm 1,\ \pm 2,\ \ldots).$$

To the matrices H(t) from the sets $\mathcal{H}_n^+, M_{\alpha,n}^+$ correspond systems whose multipliers lie on the positive real axis, to the matrices H(t) from the sets $\mathcal{H}_n^-, M_{\alpha,n}^-$ correspond systems with multipliers which lie on the negative real axis.

3^0. The set $\mathcal{O}$ decomposes into 2^k series of domains $\mathcal{O}_n^{(\mu)}$ $(\mu = \mu_1, \ldots, \mu_{2^k};\ n = 0, \pm 1, \pm 2, \ldots)$. The index μ determines the type of distribution of the multipliers of the first and second kind on the unit sphere.[19]

4^0. The set $\mathcal{H} \setminus \Gamma$ decomposes into $2(2^k - 1)$ series of domains $\mathcal{H}_n^{(\nu)}$ $(\nu = \nu_1, \ldots, \nu_{2(2^k-1)};\ n = 0, \pm 1, \pm 2, \ldots)$. The index μ determines the type of distribution of the multipliers.

5^0. In 2^0, 3^0, 4^0, the domains in one series, i.e. the domains which differ only in the index n are homeomorphic. The numbering by the index n of domains in one series obviates the statement which

[19] 3^0 is proved together with other statements in the work of I.M. Gel'fand and V.B. Lidskiy [7].

was formulated in the remark to Theorem 4.1.

To prove 1^0, we must prove in accordance with Theorems 3.2, 3.4 and 4.2 that in the set $\tilde{m}_\alpha$ and in the sets $\mathcal{H}, \tilde{M}_\alpha$ there is a closed curve V(t), when k > 1, such that

$$\operatorname{Arg} V(t)|_0^1 = 2\pi.$$

This is obvious. The curve V(t) can, for example, be defined as follows:

$$T^{-1}V(t)T = V_2(t) \oplus \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{4.7}$$

$$T^{\tau}I^{-1}T = I_2^{-1} \oplus I_2^{-1} \oplus \ldots \oplus I_2^{-1}. \tag{4.8}$$

where T is any real non-singular matrix satisfying Eq. (4.8), and the matrix $V_2(t)$ has the form

$$V_2(t) = \begin{pmatrix} \mu_1(t) & 0 \\ 0 & \mu_1(t)^{-1} \end{pmatrix}, \quad \mu_1(t) = \mu(1-3t) + 3t \quad \text{when} \quad 0 \leqslant t \leqslant \frac{1}{3},$$

$$V_2(t) = e^{I_2\varphi(t)}, \quad \varphi(t) = 6\pi\left(t - \frac{1}{3}\right) \quad \text{when} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3},$$

$$V_2(t) = \begin{pmatrix} \mu_1(t) & 0 \\ 0 & \mu_2(t)^{-1} \end{pmatrix}, \quad \mu_2(t) = 1 - 3(1-\mu)\left(t - \frac{2}{3}\right) \quad \text{when} \quad \frac{2}{3} \leqslant t \leqslant 1$$

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Then

$$\operatorname{Arg} V(t)|_0^1 = \operatorname{Arg} V_2(t)|_0^1 = \varphi(t)\Big|_{\frac{1}{3}}^{\frac{2}{3}} = 2\pi.$$

For the set $\tilde{M}_\alpha \; \mu > e^\alpha$. For the set $m_\alpha \; 1 < \mu < e^\alpha$, and for k = 1, the right members in expressions (4.7) and (4.8) have only the terms $V_2(t)$ and I_2^{-1}. For the set $\mathcal{H} \; \mu > 1$.

In conclusion, we will consider certain examples.

1. Let D be a region in the interior of the unit circle which is symmetric with respect to the real axis, and let AB be an arc on the upper semicircle. For definiteness, let $\mathfrak{M}_{AB}$ be a set of systems (0.1) for which the two multipliers ρ_1 and ρ_2 of the first kind lie on the arc AB, and the four multipliers ρ_3, ρ_4, ρ_{3*}, ρ_{4*} in the interior of the region D (see Fig. 4). We thus specify the set $\hat{\mathfrak{M}}_{AB}$ of its "projections" in Σ in terms of the set $\hat{\mathfrak{M}}_{AB}$. It is easily seen that $\hat{\mathfrak{M}}_{AB}$ is a domain. Consequently, (Theorem 2.4),

the corresponding set $\widehat{\mathfrak{M}} \subset \mathfrak{G}$ of monodromy matrices is also a domain. It is easily seen that the domain $\widehat{\mathfrak{M}}$ is simply connected in $\widetilde{\mathfrak{G}}$. This is proved in the same way as the simple connectedness of the domains $\widetilde{\mathcal{O}}^{(\mu)}$ and $\widetilde{\mathcal{H}}^{(\nu)}$ in $\mathfrak{G}$. Therefore, $\mathfrak{M}_{AB}$ decomposes into a countable number of mutually homeomorphic domains (Theorem 4.1).

We will now consider how a concrete closed curve $\zeta(t) \in \widehat{\mathfrak{M}}_{AB}$ is contracted into a point. We will first consider the movement of $\rho_1(t)$, $\rho_2(t)$ along the arc AB. If $\rho_1(0) = \rho_1(1) = \rho_1$, then $\rho_2(0) = \rho_2(1) = \rho_2$ and each curve $\rho_1(t)$, $\rho_2(t)$ can be contracted in an obvious way into a point along the arc AB. If the multiplier $\rho_1(t)$ moves from the point ρ_1 to the point ρ_2, then $\rho_2(t)$ moves from the point ρ_2 to the point ρ_1. They meet at some point n on the arc AB;because multipliers of the same kind are "indistinguishable" the movement which was described can be considered as the movement /351
of the multiplier $\rho_1(t)$ from the point ρ_1 to the point n and back to the point ρ_1, and that of the multiplier $\rho_2(t)$ from ρ_2 to n and then to ρ_2. Each of these curves can be contracted into a point: the first into the point ρ_1, the second into the point ρ_2.

The multipliers in the domain D can be treated analogously even more simply. The curve need not lie in the domain during the deformation.

If the arc AB coincides with the entire circle (or, includes one of the points ±1), the corresponding set $\widehat{\mathfrak{M}}_{AB}$ (and therefore also $\widetilde{\mathfrak{M}}_{AB}$, $\mathfrak{M}_{AB}$) is not open. In fact, now, the elements ζ' for which the corresponding multipliers lie on the unit circle can be arbitrarily close to the elements $\zeta \in \mathfrak{M}_{AB}$, for which the multipliers of different kinds coincide.

2. We will consider the set $\mathfrak{M}$ of all systems (0.1) of order twelve, for which four multipliers of the first kind lie in the domain D and two multipliers ρ_1, ρ_2 of the first kind lie in the annulus

$$0 < \alpha < |\rho_i| < \frac{1}{\alpha},$$

which, by definition, does not intersect with the domain D. The corresponding set $\hat{\mathfrak{M}}$ will clearly be open and connected. A closed curve $\zeta(t)$ exists in it such that $\operatorname{Arg}\zeta(t)|_0^1 = 2\pi$. For such a curve, we can take the curve for which (see Fig. 4) the multiplier

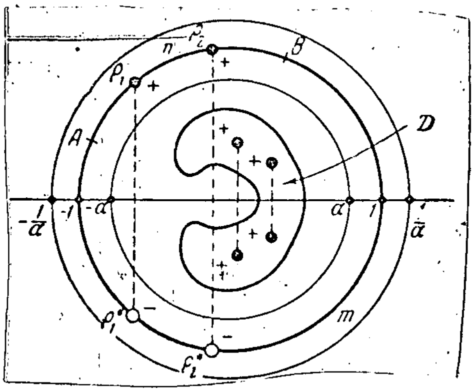

Fig. 4.

ρ_1 moves (in the counterclockwise direction) along the arc $\rho_1 m \rho_2$ of the unit circle to the point ρ_2 and the multiplier ρ_2 moves along the arc $\rho_2 m \rho_1$ to the point ρ_1. The multipliers in the domain D either do not move or pass into one another arbitrarily. Therefore, the index of the domain $\hat{\mathfrak{M}}$ (and hence also of $\hat{\mathfrak{M}}$) is all equal to one, and $\mathfrak{M}$ is a domain in $\mathfrak{H}$.

3. Let now $\mathfrak{M}$ be the set which is the same as that defined in the preceding example, with the additional condition:

$$0 < \arg\rho_1 - \arg\rho_2 < \theta < \pi, \qquad (*)$$

where θ is a fixed number. The corresponding set $\hat{\mathfrak{M}}$ is clearly a domain. Let $\zeta(t) \in \hat{\mathfrak{M}}$ be a curve for which the multipliers ρ_1 and ρ_2 having completed one revolution on the unit circle in the clockwise direction return to their original positions, satisfying, during the movement, the condition (*). The remaining multipliers, for example, need not move. Then,

$$\operatorname{Arg}\zeta(t)|_0^1 = 4\pi.$$

It is also easily seen that for any closed curve $\zeta(t) \in \hat{\mathfrak{M}}$, $\operatorname{Arg}\zeta(t)|_0^1 \geqslant 4\pi.$ Therefore, the index of the domain $\mathfrak{M}$ in $\mathfrak{G}$ (or $\hat{\mathfrak{M}}$ in Σ) is two. (Consequently, the set $\hat{\mathfrak{M}}$ lies in $\mathfrak{G}$ as shown for the set $\hat{\mathfrak{M}}_2$ in Fig. 3.) By Theorem 4.2, the set $\mathfrak{M}$ is the union of two /352
non-intersecting domains.

Infinitely many such examples which can be made arbitrarily complex can be given.

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REFERENCES

1. M.G. Kreyn, "A Generalization of Certain Studies of A.M. Lyapunov of Liner Differential Equations with Periodic Coefficients," DAN SSSR 63/3, 445-448 (1950).

2. M.G. Kreyn, "The Theory of Entire Matrix Functions of the Exponential Type," Ukr. matem. zhurn. 3/2, 163-173 (1951).

3. M.G. Kreyn, "An Application of an Algebraic Theorem in the Theory of Monodromy Matrices," Uspekhi matem. nauk 6/41, Issue 1, 171-177 (1951).

4. M.G. Kreyn, "Certain Maximum and Minimum Problems for Characteristic Numbers and Lyapunov Stability Zones," Prikl. matem. i mekh. 15/3, 323-348 (1951).

5. M.G. Kreyn, "Fundamental Postulates in the Theory of λ Stability Zones for a Canonical System of Linear Differential Equations with Periodic Coefficients," in the collection Pamyati A.A. Andronova [Memoirs of A.A. Andronov], USSR Academy of Science, Moscow, 1955, pp. 414-498.

6. M.G. Kreyn, "Stability and Boundedness Criteria for the Solutions of Periodic Canonical Systems," Prikl. matem. i mekh. 19/6, 641-680 (1955).

7. I.M. Gel'fand and V.B. Lidskiy, "The Structure of Stability Regions of Linear Canonical Systems of Differential Equations with Periodic Coefficients," Uspekhi matem. nauk 10/63, issue 1, 3-40 (1955).

8. M.G. Neygauz and V.B. Lidskiy, "Boundedness of the Solutions of Linear Systems of Differential Equations with Periodic Coefficients," DAN SSSR 64/2, 189-192 (1951).

9. V.A. Yakubovich, "The Boundedness of the Solutions of the Equation $y'' + p(t)y = 0$, $p(t + \omega) = p(t)$," DAN SSSR 64/5, 901-903 (1950).

10. V.A. Yakubovich, "Bounds for Characteristic Exponents and Stability Criteria for the Second Order Linear Differential Equation with Periodic Coefficients," DAN SSSR 64/5, 901-903 (1950).

11. V.A. Yakubovich, "Bounds for the Characteristic Exponents of a System of Linear Differential Equations with Periodic Coefficients," Prikl. matem. i mekh. 18/5, 533-546 (1954).

12. V.A. Yakubovich, "Stability of the Solutions of a System of Two Linear Differential Equations with Periodic Coefficients in Canonical Form," Matem. sb. 37/79, 21-67 (1955).

13. A.I. Mal'tsev, Osnovy lineynoy algebry [Foundations of Linear Algebra], Gostekhizdat, Moscow/Leningrad, 1948.